

AN ANALYTICAL MODEL OF PERIODIC WAVES IN SHALLOW WATER

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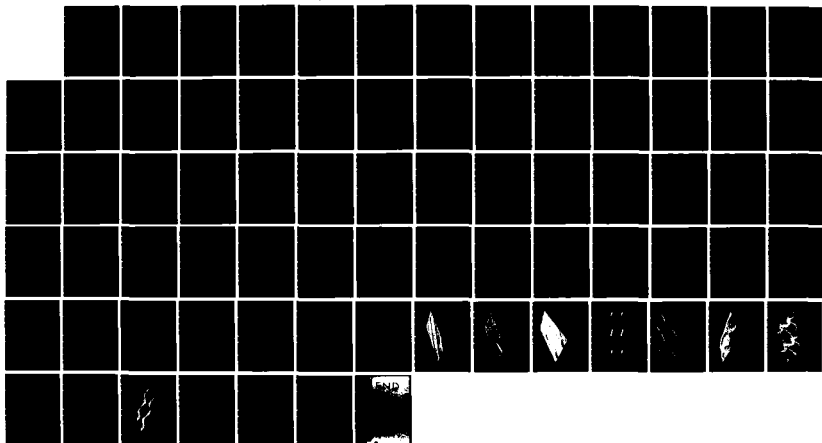
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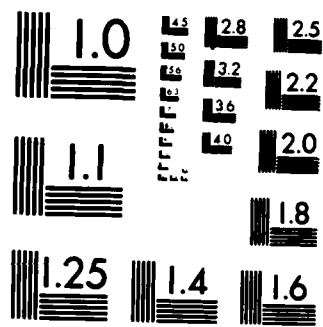
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PERIODIC WAVES IN SHALLOW WATER

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Abstract

An explicit, analytical model is presented of finite amplitude waves in shallow water. The waves in question have two independent spatial periods, in two independent horizontal directions. Both short-crested and long-crested waves are available from the model. Every wave pattern is an exact solution of the Kadomtsev-Petviashvili equation, and is based on a Riemann theta function of genus 2. These bi-periodic waves are direct generalizations of the well-known (simply periodic) cnoidal waves. Just as cnoidal waves are often used as one-dimensional models of "typical" nonlinear, periodic waves in shallow water, these bi-periodic waves may be considered to represent "typical" nonlinear, periodic waves in shallow water without the assumption of one-dimensionality.

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## 1. Introduction

In their 1895 paper, Korteweg and deVries showed that in the absence of dissipation, the evolution of unidirectional, long water waves of moderate amplitude is governed approximately by an equation equivalent to

$$f_t + 6ff_x + f_{xxx} = 0 \quad (1.1)$$

This has become the standard form for the Korteweg-deVries (KdV) equation. They also discovered periodic solutions of (1.1) in the form

$$f(x,t) = o^2 k^2 \operatorname{cn}^2 [o(x - ct + x_0); k] + f_0 \quad (1.2)$$

which they called "cnoidal waves". In (1.2),  $\operatorname{cn}[\psi; k]$  is a Jacobian elliptic function with modulus  $k$  ( $0 \leq k \leq 1$ ), the wave speed ( $c$ ) is given by

$$c = 6 f_0 - 4 o^2 (1 - 2k^2) \quad (1.3a)$$

and the wavelength ( $L$ ) is given by

$$oL = 2 K(k) \quad (1.3b)$$

where  $K(k)$  is the complete elliptic integral of the first kind (cf. Byrd & Friedman, 1971). Because (1.1) is invariant under a Galilean transformation ( $t = \tau$ ,  $x = \chi - 6 a \tau$ ,  $f(x, t) = v(\chi, \tau) + a$ ), we may normalize its periodic solutions by requiring

$$\oint f dx = 0 \quad (1.4)$$

Then in (1.2),

$$f_0 = -2\alpha^2 \left[ \frac{E(k)}{K(k)} - 1 + k^2 \right], \quad (1.5)$$

where  $E(k)$  is the complete elliptic integral of the second kind.

Two views of cnoidal waves are popular. Mathematicians tend to regard the KdV equation as a prototype of an infinite dimensional Hamiltonian system that is completely integrable. (For the analysis of (1.1) with periodic boundary conditions, see Zabusky & Kruskal (1965), Novikov (1974), Dubrovin & Novikov (1974), Dubrovin (1975), Lax (1975), Its & Matveev (1975), Flaschka (1975), McKean & van Moerbeke (1975), McKean & Trubowitz (1976).) Of particular importance is an infinite dimensional family of exact, spatially-periodic solutions of (1.1), all of which have the form

$$f(x,t) = 2\alpha_x^2 \ln \theta(\phi_1, \dots, \phi_N), \quad (1.6)$$

where  $\phi_j = \mu_j x + \omega_j t + \phi_{j0}$ , and  $\theta$  is a Riemann theta function of genus  $N$ . One may think of these "genus  $N$ " solutions of (1.1) as the periodic analogues of its  $N$ -soliton solutions, which can be obtained from (1.6) in an appropriate limit. In the simplest case, where  $N = 1$ , the solution in (1.6) is identical with that in (1.2).

Ocean engineers often adopt a different viewpoint that was popularized by Wiegel (1960). From this perspective, cnoidal waves are important not as particular solutions of a partial differential equation, but rather as one-dimensional, analytical models of "typical" nonlinear, periodic waves in shallow water. (In this paper, we use "shallow-water wave" in the following sense. Given a gravity wave with amplitude  $\zeta_0$  and wavelength  $L$ , propagating in water of depth  $h$ , its Ursell number is

$$U = \zeta_0 L^2 / h^3. \quad (1.7a)$$

We call the wave a shallow-water wave if

$$U = O(1) \quad ; \quad (1.7b)$$

see Sarpkaya & Isaacson (1981, p.215) for a detailed discussion of (1.7b). The well-known Stokes expansion of water waves requires  $U \ll 1$ , and becomes invalid in the shallow water regime of interest here.)

The practical value of this second perspective is undeniable. Jacobian elliptic functions are well-documented analytic functions, so it is fairly easy to calculate "typical" values of wave speeds, forces, mass transport, etc. for engineering purposes. Cnoidal wave theory assumes that wave amplitudes are not too large, but the theory has been generalized by Laitone (1961) and others to include higher order nonlinear effects. Sarpkaya & Isaacson (1981) summarize these generalizations, and give a concise description of cnoidal wave theory as a practical engineering tool.

One of the major shortcomings of cnoidal wave theory as a practical model of shallow-water waves is that the theory is one-dimensional, whereas the water surface is two-dimensional. It follows that cnoidal waves are necessarily long-crested, whereas both long-crested and short-crested waves are observed in shallow water.

The objective of this paper is to describe a two-dimensional generalization of cnoidal waves. The waves to be described are periodic in two independent directions on the water surface, and they are proposed here as models of two-dimensional, periodic waves of finite amplitude in shallow water. (There is always a semantic confusion about counting dimensions in these problems. Cnoidal waves have two-dimensional velocity fields and one-dimensional surface patterns; in this paper, we call cnoidal waves one-dimensional. The waves of interest here have three-dimensional velocity fields, and two-dimensional surface patterns; we call them two-dimensional.) To give the reader some notion of what this model actually is and in what sense it generalizes the cnoidal wave, we now list some of its main features. This list should be interpreted as a series of claims, which will be verified

in the body of the paper.

- a) Just as the KdV equation describes the evolution of long water waves of finite amplitude if they are strictly one-dimensional, the KP equation,

$$\left(f_t + 6ff_x + f_{xxx}\right)_x + 3f_{yy} = 0 \quad , \quad (1.8)$$

describes their evolution if they are weakly two-dimensional (Kadomtsev & Petviashvili, 1970). The evolution described by (1.8) is weakly nonlinear, weakly dispersive, weakly two-dimensional, and all three effects are permitted to enter at the same order.

- b) Both the KdV and KP equations are Galilean invariant, so their periodic solutions may be normalized by imposing a condition like (1.4).  
c) Like the KdV equation, the KP equation admits an infinite-dimensional family of exact solutions in the form

$$f(x,y,t) = 2\partial_x^2 \ln \theta \quad , \quad (1.9)$$

where  $\theta$  is a Riemann theta function of genus  $N$ . If  $N = 1$ , these solutions are simply cnoidal waves.

- d) The waves of interest in this paper appear at  $N = 2$ . Then the KP equation admits solutions in the form (1.9) that are periodic in two independent horizontal directions. Examples of these two-dimensional waveforms are shown in Figures 2 and 3. In this paper, we will refer to these waves as "bi-periodic," or as "KP solutions of genus 2." (To avoid confusion we emphasize that our bi-periodic waves have two real periods. They should not be confused with ordinary elliptic functions, which have two complex periods but only one real period.)  
e) The cnoidal wave in (1.2) is stationary if one travels with the speed defined by (1.3). The KP solutions of genus 2 that are of interest here also are stationary if one travels with the correct (constant) velocity. In both cases, there is only one wave pattern for all time. For cnoidal waves the surface pattern is one-dimensional; for bi-periodic waves it is two-dimensional.  
f) The cnoidal waves in (1.2) have three arbitrary parameters  $(k, \phi, x_0)$ , of



which two  $(k, \omega)$  have dynamical significance. These two parameters can be determined uniquely from two appropriate measurements of the wave. The bi-periodic waves have eight arbitrary parameters, of which six have dynamical significance. We will show how these six parameters can be determined directly from a few measurements of the wave.

- g) The derivations that lead to both the KdV and KP equations assume that the waves in question are too small to break. Neither equation ever predicts breaking waves, but either equation may predict waves of large amplitude that may break according to some other criterion.

The organization of this paper is as follows. In §2 is a brief derivation of the KP equation from the equations for inviscid water waves. KP solutions of genus 1 (cnoidal waves) are reviewed in §3. The KP solutions of genus 2 are the main concern of this paper, and these are described in detail in §4. How to use these solutions as models of bi-periodic waves in shallow water is described in §5.

## 2. Physical meaning of the KP equation

The KP equation is a two-dimensional generalization of the KdV equation, and its derivation as an approximate model for water waves follows closely that of the KdV equation. Because the derivation has been given elsewhere (e.g., Ablowitz & Segur, 1979), we merely review the main points here.

In the simplest case, we seek the irrotational motion of an inviscid, incompressible, homogeneous fluid, subject to a constant (vertical) gravitational force,  $g$ . We employ Cartesian coordinates  $(\chi, \eta, \xi)$  in which  $\xi$  increases upward. The fluid rests on a horizontal, impermeable bed at  $\xi = -h$  and has a free surface at  $\xi = \zeta(\chi, \eta, \tau)$ . Because the motion is irrotational by assumption, there is a velocity potential  $\phi$  that satisfies

$$\nabla^2 \phi = 0, \quad -h < \xi < \zeta(\chi, \eta, \tau) \quad . \quad (2.1)$$

It is subject to boundary conditions on the bottom ( $\xi = -h$ ),

$$\partial_{\xi} \phi = 0 \quad , \quad (2.2a)$$

and along the free surface ( $\xi = \zeta(\chi, \eta, \tau)$ ),

$$\frac{D\zeta}{D\tau} = \zeta_{\tau} + \phi_{\chi} \zeta_{\chi} + \phi_{\eta} \zeta_{\eta} = \phi_{\xi} \quad , \quad (2.2b)$$

$$\phi_{\tau} + g\zeta + \frac{1}{2} |\nabla \phi|^2 = 0 \quad . \quad (2.2c)$$

In this discussion, we are interested in spatially periodic solutions of (2.1), (2.2), and we require

$$\lim_{L \rightarrow \infty} \int_0^L d\chi \zeta(\chi + \chi_0, \eta, T) = 0 \quad , \quad (2.3)$$

This is a normalization of  $\zeta$ , corresponding to (1.4). It implies that  $h$  is the mean fluid depth, and that  $\zeta = 0$  in the absence of any motion.

Let  $\vec{\kappa} = (\ell, m)$  be a representative horizontal wave number for the waves in question, with  $\kappa^2 = \ell^2 + m^2$ . Orient the horizontal coordinates so that the  $x$ -direction is the principal direction of wave propagation. To derive the KP equations from (2.1) and (2.2), we make four assumptions.

(A) Wave amplitudes are small:

$$\epsilon = \frac{|\zeta|_{\max}}{h} \ll 1.$$

(B) The water is shallow relative to typical horizontal wavelengths:

$$(\kappa h)^2 \ll 1.$$

(C) The waves are nearly one-dimensional:

$$(m/\ell)^2 \ll 1.$$

(D) These three small effects all have comparable influence:

$$(m/\ell)^2 = O[(\kappa h)^2] = O(\epsilon).$$

When these assumptions are used to re-scale the original equations systematically, the result at leading order is a one-dimensional wave equation. It follows that the surface displacement is given by

$$\zeta(x, \eta, \tau) = \epsilon h \left[ f(x - \sqrt{gh} \tau; \eta, \tau) + F(x + \sqrt{gh} \tau; \eta, \tau) \right] + O(\epsilon^2 h)$$

where  $f$  and  $F$  are known in terms of initial data, and each varies slowly in its last two variables. The expansion procedure can be implemented in terms of the following scaled, dimensionless variables,

$$x = \epsilon^{1/2}(\chi - (gh)^{1/2}\tau)/h, \quad \bar{x} = \epsilon^{1/2}(\chi + (gh)^{1/2}\tau)/h, \quad (2.4a)$$

$$y = \epsilon \eta / h,$$

$$t = \epsilon^{3/2} (gh)^{1/2} \tau / 6h.$$

Then we look for solutions of (2.1), (2.2) of the form

$$\zeta = \frac{2\epsilon h}{3} \left[ f(x, y, t) + F(\bar{x}, y, t) \right], \quad (2.4b)$$

etc. Secular terms arise at second-order in this expansion unless the "right-going" waves satisfy the KP equation,

$$(f_t + 6f f_x + f_{xxx})_x + 3f_{yy} = 0, \quad (1.8)$$

and the "left-going" waves satisfy an equivalent equation. To make the model quite explicit, we also write the KP equation for the right-going waves in its dimensional form for  $\zeta(x, \eta, \tau)$ :

$$\left[ (gh)^{-1/2} \zeta_\tau + \zeta_x + \frac{3}{2} \frac{\zeta}{h} \zeta_x + \frac{h^2}{6} \zeta_{xxx} \right]_x + \frac{1}{2} \zeta_{\eta\eta} = 0 \quad (2.5)$$

The components of fluid velocity in the  $(x, \eta, \xi)$  directions are, for  $-h \leq \xi \leq \zeta(x, \eta, \tau)$ ,

$$u = (g/h)^{1/2} \zeta, \quad (2.6a)$$

$$v = (g/h)^{1/2} \int^x d\chi (\partial_\eta \zeta), \quad (2.6b)$$

$$w = -(g/h)^{1/2} (h + \xi) \partial_x \zeta. \quad (2.6c)$$

Note that only the vertical component of velocity ( $w$ ) has any vertical structure at leading order. To the same order of approximation, the fluid pressure below the free surface is simply hydrostatic:

$$p = \rho g(\zeta - \xi). \quad (2.6d)$$

Some comments about this derivation of the KP equation:

- (i) Surface tension was omitted for simplicity. The effect of including it is to change the numerical coefficient of  $(\zeta_{xxxx})$  in (2.5). For water beneath air, this change is negligible if the water depth exceeds a few centimeters. If the water depth is less than about 1/2 cm, however, this coefficient changes sign and this changes the qualitative character of the solution of the KP equation. We recommend caution when using the model described here for periodic waves on thin films of liquid.
- (ii) This derivation neglects dissipation, which is assumed to be even weaker than the weak nonlinearity, weak dispersion and weak two-dimensionality included in the KP equation.
- (iii) The assumption that the motion is irrotational is convenient but not necessary, as shown by Benney (1966).
- (iv) The original equations (2.1, 2.2) are invariant under a horizontal rotation of coordinates. This symmetry is broken in choosing the  $x$ -direction as the principal direction of wave propagation. Even so, the KP equation retains a remnant of this symmetry; (1.8) is invariant under transformations of the form:

$$\begin{aligned}
 x &\rightarrow x + \alpha y - 3\alpha^2 t, & \partial_x &\rightarrow \partial_x, \\
 y &\rightarrow y - 6\alpha t, & \partial_y &\rightarrow \partial_y + \alpha \partial_x, \\
 t &\rightarrow t, & \partial_t &\rightarrow \partial_t - 6\alpha \partial_y - 3\alpha^2 \partial_x.
 \end{aligned}
 \tag{2.7}$$

We will refer to these transformations loosely as "rotations," but we emphasize that this symmetry group only approximates actual rotations in the physical  $(x, y)$  plane, and only if  $|\partial_x| \gg |\partial_y|$ .

Thus, the KP equation admits waves that travel in arbitrary directions in the  $(x, y)$  plane, but we can expect them to model water waves

accurately only if they propagate primarily in the x-direction.

- (v) The KP equation also exhibits a scaling symmetry.

$$x \rightarrow \beta x, \quad t \rightarrow \beta^3 t, \quad y \rightarrow \beta^2 y, \quad f \rightarrow \beta^{-2} f \quad . \quad (2.8)$$

This symmetry is inherited from the arbitrariness of the small parameter ( $\epsilon$ ) in (2.4). The symmetry parameters, ( $\alpha, \beta$ ), are two of the six parameters available in the model of doubly periodic waves in shallow water.

- (vi) We mentioned below (2.3) that (2.3) defines  $h$  uniquely. Because of (2.6a), (2.3) also identifies the laboratory coordinate system (i.e.,  $x, n, \xi, \tau$ ). Integrating (2.3) over  $\int_{-h}^{\xi} d\xi$  shows that the mean horizontal motion vanishes in the  $x$ -direction, so there is no net mass transport in that direction in the laboratory frame.

### 3. KP solutions of genus 1 - cnoidal waves

Cnoidal waves have been discussed at length elsewhere. (For a hydrodynamic emphasis, see Wiegel, 1960 or Sarpkaya & Isaacson, 1981; for a mathematical emphasis, see Dubrovin, 1981 or Boyd, 1982.) We review them here because the bi-periodic waves are direct generalizations of cnoidal waves.

#### 3.1 Construction of KP solutions of genus 1

Real-valued KP solutions of genus 1 are constructed as follows. The phase variable is

$$\phi = \mu x + \nu y + \omega t + \phi_0, \quad (3.1a)$$

or

$$\phi = \mu (x + py - ct) + \phi_0; \quad (3.1b)$$

$\phi$  is required to be real. A theta function of genus 1 is defined by a Fourier series:

$$\theta(\phi; b) = \sum_{n=-\infty}^{\infty} \exp\left(\frac{1}{2} b n^2 + i n \phi\right) \quad (3.2)$$

$\theta$  is both convergent and real-valued for all real  $\phi$  if  $b$  is a negative real number, which we now require. (Differences in notation are always a nuisance.  $\theta(\phi)$  and  $b$  are called  $\vartheta_3(\phi/2)$  and  $(2\pi i\tau)$ , respectively, by Whittaker and Watson, 1927.) Krichever (1976) showed that the KP equation (1.8) has

solutions of the form

$$f(x, y, t) = 2\alpha_x^2 \ln \theta(\phi; b) \quad , \quad (3.3)$$

provided  $c$  is a particular function of  $(\mu, \rho, b)$ . Standard identities for elliptic functions (cf. Byrd and Friedman, 1971) show that (3.3) is equivalent to

$$f = 2 \left( \frac{\mu K(k)}{\pi} \right)^2 \left[ k^2 \operatorname{cn}^2(\psi; k) + 1 - k^2 - \frac{E(k)}{K(k)} \right] \quad , \quad (3.4a)$$

where

$$\psi = \frac{K(k)}{\pi} (\phi - \pi) \quad , \quad (3.4b)$$

$$b = -2\pi K \left( \left[ 1 - k^2 \right]^{1/2} \right) / K(k) \quad . \quad (3.4c)$$

If  $\rho = 0$  in (3.1), then (3.4) is identical with (1.2) and (1.5), the formulae for a cnoidal wave solution of the KdV equation. Its speed is given by (1.3a) with  $c = \mu K(k)/\pi$ . For  $\rho \neq 0$ , we may use (2.7) with  $\alpha = \rho$  to rotate this KdV solution into the general real-valued KP solution of genus 1, with the following properties.

- i) There are four arbitrary real parameters in the solution:  $(\mu, \rho, \phi_0)$  in (3.1), and  $b$  ( $b < 0$ ) in (3.2). Of these,  $\mu$  represents the scaling symmetry in (2.8),  $\rho$  represents the rotational symmetry in (2.7), and  $\phi_0$  represents the translational symmetry of any of the independent variables in (1.8). There is no dynamical significance to  $\phi_0$ .
- ii) Given these parameters,  $k$  is uniquely determined by (3.4c), the waveform is given by (3.3) or (3.4a), and the speed parameter is given by

$$c = 3\rho^2 - \left( \frac{2\mu K(k)}{\pi} \right)^2 \left[ 3 \frac{E(k)}{K(k)} - 2 + k^2 \right] \quad . \quad (3.5)$$

An equivalent expression for the speed in terms of  $(\mu, \rho, b)$  was given by Dubrovin (1981).

- iii) The wavelength, direction of propagation and total speed all follow from (3.1) and elementary geometric considerations. A typical cnoidal wave is shown in Figure 1.



Instead of (3.2), we may also write

$$\theta(\phi; b) = \left(\frac{2\pi}{-b}\right)^{1/2} \sum_{m=-\infty}^{\infty} \exp\left[\frac{2}{b}\left(\frac{\phi}{2} - m\pi\right)^2\right]. \quad (3.6)$$

This representation follows from an identity known as the "Poisson sum formula,"

$$\sum_{n=-\infty}^{+\infty} \exp(n^2 s + 2niz) = \left(-\frac{\pi}{s}\right)^{1/2} \sum_{m=-\infty}^{\infty} \exp\left[\frac{(z - m\pi)^2}{s}\right], \quad (3.7)$$

which is valid for  $\text{Re}(s) < 0$  (Whittaker & Watson, 1927, p. 476). For any  $b < 0$ , it follows from (3.6) that the two terms corresponding to  $m = 0$  and  $m = +1$  are larger than any others in the neighborhood of  $\phi = \pi$ . If  $b \rightarrow 0$ , then in this region these two terms are even dominant in an asymptotic sense. [Similarly, the terms corresponding to  $M$  and  $(M+1)$  dominate in the neighborhood of  $\phi = (2M+1)\pi$ . Thus each pair of terms in (3.6) dominates over exactly one period of  $\theta(\phi)$ .] Retaining only the terms corresponding to  $m = 0, 1$  in (3.6) and substituting into (3.3) yields the expression for single KP soliton riding on a depressed mean level (at  $2\mu^2/b$ ):

$$\begin{aligned} f(x, y, t) &\sim 2\partial_x^2 \ln\left[1 + \exp\left\{\frac{-2\pi}{b}(\phi - \pi)\right\}\right] + 2\mu^2/b \\ &\sim 2\left(\frac{\pi\mu}{b}\right)^2 \text{sech}^2\left[\frac{\pi}{b}(\phi - \pi)\right] + \frac{2\mu^2}{b}. \end{aligned} \quad (3.8)$$

This is correct asymptotically if  $b \rightarrow 0$  with  $(\mu/b)$  finite. It follows that the cnoidal wave can be viewed as a superposition of overlapping solitary waves, placed one period apart. This view has been discussed at length by Boyd (1982).

### 3.2 Reconstructing a cnoidal wave from simple measurements

In order to use the KP solutions of genus 1 as models of periodic waves in shallow water, one must be able to infer the parameters ( $\mu$ ,  $\rho$ ,  $b$ ) from measurements of the wave. This can be done in a variety of ways, one of which is outlined here.

The cnoidal waves have a simple characterization.

- i) A cnoidal wave is real-valued and bounded, for all real ( $x$ ,  $y$ ,  $t$ ).
- ii) It is one-dimensional; i.e., there is a single phase variable [ $\phi$  in (3.1)], which is linear in ( $x$ ,  $y$ ,  $t$ ). There is no variation along the wave crests (i.e., along  $\phi = \text{const}$ ). In the terminology of oceanography, cnoidal waves are long-crested.
- iii) A cnoidal wave is periodic (in  $\phi$ ).
- iv) There is a uniformly translating coordinate system in which the cnoidal wave is stationary.

Cnoidal waves are the only KP solutions with these four properties.

Given a shallow-water wave with these four properties, one can determine the parameters of the corresponding cnoidal wave in three steps, as follows.

- a) Depending on the problem, one may or may not be able to orient the  $x$  - axis in the direction of wave propagation. If so, then  $\rho = 0$  in (3.1) and in (3.5). If not, then  $\rho$  measures the angle between these two directions; one can expect discrepancies between the cnoidal wave model and the water wave if  $\rho$  is large. In any case,  $\rho$  can be measured directly from the wave pattern.
- b) Once  $\rho$  is known,  $\mu$  is determined from  $\rho$  and  $L$ , the wavelength in the direction of wave propagation:

$$\mu = 2\pi \left[ L^2 (1 + \rho^2) \right]^{-1/2} . \quad (3.9)$$

- c) Given  $\mu$  and  $\rho$ ,  $b$  can be determined from the wave amplitude or from the speed, either of which can be measured directly. Here we focus on the speed. Suppose  $\mu$ ,  $\rho$  and  $c$  have been determined by direct measurement. From (3.5), define

$$\psi = -\frac{c - 3\rho^2}{\mu^2} = \left(\frac{2K(k)}{\pi}\right)^2 \left[ -3 \frac{E(k)}{K(k)} + 2 - k^2 \right]. \quad (3.10)$$

One can prove that  $\psi$  is a monotonic function of  $k$ , and also of  $b$ . Therefore,  $b$  is determined from  $\psi$ . Simple approximations are obtained by expanding (3.4c) and (3.10) for  $k \rightarrow 1$  and for  $k \rightarrow 0$ . The results are as follows. As  $b \rightarrow -\infty$  ( $k \rightarrow 0$ ),

$$\psi \sim -1 + 24e^b. \quad (3.11a)$$

As  $b \rightarrow 0$  ( $k \rightarrow 1$ ),

$$\psi \sim \left(\frac{2\pi}{-b}\right)^2 \left(1 + \frac{3b}{\pi^2}\right). \quad (3.11b)$$

For  $-\infty < b < 0$ ,

$$-1 + 24e^b < \psi < \left(\frac{2\pi}{-b}\right)^2 \left(1 + \frac{3b}{\pi^2}\right). \quad (3.11c)$$

Moreover, one may approximate  $\psi$  by (3.11a) for  $b \leq -4.6$ , and by (3.10b) for  $b \geq -4.6$ , with an error that never exceeds 1%. This is adequate for comparisons with most experimental data.

#### 4. KP solutions of genus 2

The KP solutions of primary interest in this paper have genus 2. They are two-dimensional generalizations of cnoidal waves. Aspects of these solutions have been considered by Krichever (1976), Nakamura (1979), Krichever and Novikov (1980), and Dubrovin (1981). Related work by Hirota and Ito (1981), Bryant (1982), and Finkel and Segur (1984) is also of interest. The work of Dubrovin (1981) is particularly important from our perspective.

##### 4.1 Construction of theta functions of genus 2

The construction of real-valued, nonvanishing theta functions of genus 2 requires a certain amount of machinery, which we now introduce. The first ingredient is a two-component vector of phase variables,  $\vec{\phi} = (\phi_1, \phi_2)^T$ , where

$$\phi_j = \mu_j x + \nu_j y + \omega_j t + \phi_{j0}, \quad j = 1, 2 \quad (4.1a)$$

$$= \mu_j (x + \rho_j y - c_j t) + \phi_{j0}, \quad j = 1, 2 \quad (4.1b)$$

Both components are required to be real. To change these variables into those of Dubrovin, let  $\mu_j \rightarrow iU_j$ ,  $\nu_j \rightarrow iV_j$ ,  $\omega_j \rightarrow -4iW_j$ ,  $j = 1, 2$ .

The second ingredient is a real-valued Riemann matrix: a 2x2, symmetric matrix that is real-valued and negative definite. Let

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{pmatrix} \quad (4.2a)$$

be real-valued.  $B$  is negative definite if

$$b_{11} < 0, \quad b_{22} < 0, \quad b_{11} b_{22} - b_{12}^2 > 0 \quad (4.2b)$$

A theta function of genus 2 is defined by a double Fourier series:

$$\theta(\phi_1, \phi_2, B) = \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} \exp \left\{ \frac{1}{2} \vec{m} \cdot B \cdot \vec{m} + i \vec{m} \cdot \vec{\phi} \right\}, \quad (4.3)$$

where

$$\vec{m} \cdot B \cdot \vec{m} = m_1^2 b_{11} + 2 m_1 m_2 b_{12} + m_2^2 b_{22} \quad ,$$

$$\vec{m} \cdot \vec{\phi} = m_1 \phi_1 + m_2 \phi_2 \quad .$$

Clearly  $\theta$  is real-valued. It converges uniformly in  $\phi_1, \phi_2$  because of (4.2b). It is periodic in  $\phi_1$  when  $\phi_2$  is held fixed, and vice-versa (i.e., it is quasi-periodic). We show in § 4.3 that  $\theta$  does not vanish.

This description needs further refinement for two reasons. First, two different Riemann matrices can be equivalent, in the sense that they generate the same theta function. For example, let

$$B_1 = \begin{pmatrix} b & b\lambda \\ b\lambda & b\lambda^2 + d \end{pmatrix}, \quad B_2 = \begin{pmatrix} b & b(\lambda+1) \\ b(\lambda+1) & b(\lambda+1)^2 + d \end{pmatrix}.$$

If  $b$  and  $d$  are both negative and  $\lambda$  is real, then  $B_1$  and  $B_2$  are both Riemann matrices. Let

$$\vec{m} = (m_1, m_2), \quad \vec{n} = [(m_1 + m_2), m_2] \quad .$$

One shows by direct computation that

$$\vec{m} \cdot B_2 \cdot \vec{m} = \vec{n} \cdot B_1 \cdot \vec{n} \quad .$$

Then it follows (by renumbering in (4.3)) that  $B_1$  and  $B_2$  produce the same theta function, and are equivalent. Any two equivalent Riemann matrices can be related to each other by a symplectic transformation (see Dubrovin, 1981, for details).

To eliminate this ambiguity, Finkel and Segur (1984) introduce the idea of a basic Riemann matrix.

Definition: A real-valued, 2x2 Riemann matrix is said to be basic if

$$-\frac{1}{2} < \lambda_j \leq \frac{1}{2}, \quad j = 1, 2, \quad (4.4)$$

where  $\lambda_j = b_{12}/b_{jj}$ . They also prove the following results:

- i) Every real-valued, 2x2 Riemann matrix is equivalent to a basic Riemann matrix.
- ii) Two basic Riemann matrices are equivalent only if their  $\lambda$ 's coincide.

It follows that every real theta function of genus 2 can be generated from a basic Riemann matrix. Henceforth, we consider only basic Riemann matrices, since any other real Riemann matrix can be reduced to its basic form via an algorithm given by Finkel & Segur (1984).

Basic Riemann matrices have a natural representation as follows. Define

$$\begin{aligned} b &= \max (b_{11}, b_{22}) \\ \lambda &= b_{12}/b \\ d &= \det B / b. \end{aligned} \quad (4.5)$$

Then every basic Riemann matrix takes one of two forms

$$B_1 = \begin{pmatrix} b & b\lambda \\ b\lambda & b\lambda^2 + d \end{pmatrix} \text{ or } B_2 = \begin{pmatrix} b\lambda^2 + d & b\lambda \\ b\lambda & b \end{pmatrix}. \quad (4.6a)$$

With

$$b < 0, \lambda^2 \leq \frac{1}{2}, \quad d \leq b(1 - \lambda^2). \quad (4.6b)$$

The relation between these two forms is

$$\theta(\phi_1, \phi_2; B_1) = \theta(\phi_2, \phi_1; B_2). \quad (4.6c)$$

For definiteness, we will always use  $B_1$  unless otherwise noted.

The second difficulty with (4.3) is that it admits degenerate cases. A 2x2 Riemann matrix is called decomposable if it is equivalent to a diagonal matrix, and indecomposable otherwise. For example, in (4.2)  $B$  is decomposable if  $b_{12} = 0$ . In this case  $\theta$  in (4.3) becomes a product of two theta functions of genus 1. Dubrovin (1981) shows that theta functions of genus 2 generate nontrivial KP solutions if and only if they are indecomposable. He also gives an explicit test for decomposability, which requires the evaluation of a 4x4 determinant. A simpler test is given by the following Theorem (Finkel and Segur, 1984): Let  $B$  be a basic Riemann matrix, as in (4.6). Then  $B$  is decomposable if and only if  $\lambda = 0$ .

#### Summary:

Let  $(b, d, \lambda)$  be three real parameters satisfying (4.6b), with  $\lambda \neq 0$ . These parameters generate a basic Riemann matrix in the form (4.6a). Every real, 2x2, indecomposable Riemann matrix is equivalent to a basic Riemann matrix. Basic Riemann matrices with  $\lambda \neq 0$  generate real-valued, indecomposable theta functions. These can be written in the form

$$\theta(\phi_1, \phi_2; B) = \sum_{m_2=-\infty}^{\infty} \exp \left\{ \frac{1}{2} d m_2^2 \right\} \sum_{m_1=-\infty}^{\infty} \exp \left\{ \frac{1}{2} b (m_1 + \lambda m_2)^2 \right\} \cos \left\{ m_1 \phi_1 + m_2 \phi_2 \right\}, \quad (4.7)$$

which follows from (4.3) and (4.6a).



## 4.2 Construction of KP solutions of genus 2

Given an indecomposable theta function of genus 2, KP solutions are constructed in the usual way:

$$f(x,y,t) = 2 \partial_x^2 \ln \theta(\phi_1, \phi_2, B). \quad (4.8)$$

Dubrovin (1981) proves that  $f(x,y,t)$  actually satisfies the KP equation, provided that its parameters satisfy certain relations. To express these relations, we need two more concepts: theta-constants, and two more phase variables.

Let  $\vec{p}$  be a two component vector, which can take on one of four values:

$$\vec{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}, \quad (4.9)$$

Every Riemann matrix generates a four-component theta constant of the form

$$\hat{\theta}[\vec{p}] = \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} \exp \left\{ (\vec{m} + \vec{p}) \cdot B \cdot (\vec{m} + \vec{p}) \right\}, \quad (4.10a)$$

where  $\vec{m} = (m_1, m_2)$ , and the product is defined below (4.3). If  $B$  is in basic form (for definiteness,  $B_1$  in (4.6a)) then this can be written as

$$\hat{\theta}[\vec{p}] = \sum_{m_2} \exp \left\{ d (m_2 + p_2)^2 \right\} \sum_{m_1} \exp \left\{ b (m_1 + p_1 + \lambda (m_2 + p_2))^2 \right\}, \quad (4.10b)$$

Obviously,  $\hat{\theta}[\vec{p}]$  is differentiable with respect to  $(b, d \text{ and } \lambda)$ , and each of these derivatives is a 4-component vector indexed by  $\vec{p}$ . Define

$$\phi_4 = \phi_2 - \lambda \phi_1, \quad \phi_3 = \phi_1 - \bar{\lambda} \phi_2, \quad (4.11a)$$

where

$$\bar{\lambda} = \frac{b_{12}}{b_{22}} = \frac{b\lambda}{b\lambda^2 + d}. \quad (4.11b)$$

We show in § 4.3 that the wave crests lie on  $\phi_3 = \text{const}$  and on  $\phi_4 = \text{const}$ . The corresponding wave numbers are

$$\mu_3 = \mu_1 - \bar{\lambda} \mu_2, \quad \mu_4 = \mu_2 - \lambda \mu_1 \quad (4.11c)$$

with similar definitions for  $(\nu_3, \omega_3, \nu_4, \omega_4)$ .

Now we may state the main Theorem (Dubrovin, 1981): Let  $\mathbb{B}$  be a real, indecomposable Riemann matrix; i.e., satisfying (4.6) with  $\lambda \neq 0$ . Then a function of the form (4.8) solves the KP equation (1.8) if and only if its parameters are related by the four equations, indexed by  $\vec{p}$ , in (4.12).

$$\mathbb{M} \vec{\chi} = 4 \mathbb{S} \vec{\psi}, \quad (4.12)$$

where

$$\vec{\chi} = \begin{pmatrix} \mu_1 \omega_1 + 3\nu_1^2 \\ \mu_1 \omega_4 + \mu_4 \omega_1 + 6\nu_1 \nu_4 \\ \mu_4 \omega_4 + 3\nu_4^2 \\ D \end{pmatrix}, \quad \vec{\psi} = \begin{pmatrix} \mu_1^4 \\ 4\mu_1^3 \mu_4 \\ 6\mu_1^2 \mu_4^2 \\ 4\mu_1 \mu_4^3 \\ \mu_4^4 \end{pmatrix},$$

$$(\mathbb{M} = a_b \hat{\theta}[\vec{p}], \frac{1}{2b} a_\lambda \hat{\theta}[\vec{p}], a_d \hat{\theta}[\vec{p}], \hat{\theta}[\vec{p}]).$$

$$\underline{S} = \left( \partial_b^2 \hat{\theta}[\hat{p}], \partial_b \left( \frac{1}{2b} \partial_\lambda \right) \hat{\theta}[\hat{p}], \partial_b \partial_d \hat{\theta}[\hat{p}], \partial_d \left( \frac{1}{2b} \partial_\lambda \right) \hat{\theta}[\hat{p}], \partial_d^2 \hat{\theta} \right),$$

and  $D$  is a constant of integration with no physical significance.  $\underline{M}$  is invertible if and only if  $\underline{B}$  is indecomposable, so (4.12) always can be solved for  $\underline{X}$ .

After  $\underline{M}$  is inverted in (4.12), the first and third of the resulting equations can be written as

$$\psi_1 = - \frac{c_1 - 3 \rho_1^2}{\mu_1^2} = \frac{\mu_1 \omega_1 + 3 v_1^2}{\mu_1^4} = P_1 \left( \frac{\mu_4}{\mu_1}; 4 \right), \quad (4.13a)$$

$$\psi_4 = - \frac{c_4 - 3 \rho_4^2}{\mu_4^2} = \frac{\mu_4 \omega_4 + 3 v_4^2}{\mu_4^4} = \left( \frac{\mu_1}{\mu_4} \right)^4 P_4 \left( \frac{\mu_4}{\mu_1}; 4 \right). \quad (4.13b)$$

Here  $P_1(z;4)$  and  $P_4(z;4)$  are well-defined polynomials of degree 4. Then eliminating  $(D, \omega_1, \omega_4)$  from (4.12) yields one more relation,

$$\left( \mu_1 v_4 - \mu_4 v_1 \right)^2 = \mu_1^6 P \left( \frac{\mu_4}{\mu_1}; 6 \right), \quad (4.13c)$$

where  $P(z;6)$  is a well-defined polynomial of degree 6. The resulting solution will be real-valued if  $\mu_4/\mu_1$  is chosen so that

$$P \left( \mu_4/\mu_1; 6 \right) \geq 0. \quad (4.13d)$$

To summarize, (4.8) provides a real-valued solution of the KP equation, (1.8), whenever: (i)  $\underline{B}$  is indecomposable (i.e.,  $\lambda \neq 0$  in (4.6)); and (ii) (4.13) is satisfied. There are no other constraints.

The conditions in (4.13) can be interpreted in terms of the wave pattern as follows:

(i) According to (4.11c),

$$\left( \mu_1 v_2 - \mu_2 v_1 \right)^2 = \left( \mu_1 v_4 - \mu_4 v_1 \right)^2 = \left( \mu_3 v_2 - \mu_2 v_3 \right)^2. \quad (4.14)$$

(ii. If  $\mu_1 v_2 = \mu_2 v_1$ , then at any fixed  $t$ ,  $\phi_1 = \text{constant}$  and  $\phi_2 = \text{constant}$  on parallel lines in the  $(x,y)$ -plane. Using (2.7), such a solution can be "rotated" so that  $v_1 = v_2 = 0$ ; i.e., these are actually KdV solutions of genus 2. According to (4.13c) and (4.14),  $\mu_1 v_2 = \mu_2 v_1$  wherever  $P(\mu_4/\mu_1; 6) = 0$ . In this way, every indecomposable  $2 \times 2$  Riemann matrix generates exactly 6 KdV solutions of genus 2, corresponding to the 6 roots of  $P(\mu_4/\mu_1; 6) = 0$ . The solutions may or may not be real. This identification is due to Dubrovin (1981). Our interest in the present paper is in two-dimensional waves, so we will systematically neglect these one-dimensional KdV solutions.

(iii) Suppose  $\mu_1 v_2 \neq \mu_2 v_1$ . Then

$$\iint d\phi_1 \wedge d\phi_2 = (\mu_1 v_2 - \mu_2 v_1) \iint dx \wedge dy \quad (4.15)$$

represents a non-zero area-element in the  $(x,y)$ -plane. For any fixed  $(\bar{\phi}_1, \bar{\phi}_2)$ , a period parallelogram of  $\theta(\phi_1, \phi_2; \mathbb{R})$  has as its vertices four points defined by

$$(\phi_1, \phi_2) = (\bar{\phi}_1, \bar{\phi}_2), (\bar{\phi}_1 + 2\pi, \bar{\phi}_2), (\bar{\phi}_1 + 2\pi, \bar{\phi}_2 + 2\pi), (\bar{\phi}_1, \bar{\phi}_2 + 2\pi).$$

The area,  $A$ , of this parallelogram is given by

$$\int_0^{2\pi} \int_0^{2\pi} d\phi_1 \wedge d\phi_2 = (\mu_1 v_2 - \mu_2 v_1) \iint dA,$$

so that

$$\left( \mu_1 \nu_2 - \mu_2 \nu_1 \right)^2 = \frac{(2\pi)^4}{A^2} . \quad (4.16)$$

It follows that (4.13c) can be interpreted as specifying the area of the period parallelogram of each KP solution. As one might guess, this area is independent of rotations (2.7), but depends on scaling (2.8).

(iv) If one seeks a "nonlinear dispersion relation" for these waves of genus 2, it is given by (4.13a,b). These two relations are the generalization to genus 2 of (3.10).  $\Psi_1$  and  $\Psi_4$  are independent of either of the symmetries in (2.7) or (2.8). They depend only on  $B$  and on  $(\mu_4/\mu_1)$ . They determine the wave speeds.

(v) Suppose  $\mu_1 \nu_2 \neq \mu_2 \nu_1$ . Then at any  $t$ ,  $\phi_1 = \text{constant}$  and  $\phi_2 = \text{constant}$  intersect transversely at a unique point  $(x(t), y(t))$ . As  $t$  changes, this point moves in the  $(x,y)$ -plane with a constant speed given by

$$\frac{dx}{dt} = \frac{\nu_1 \omega_2 - \nu_2 \omega_1}{\mu_1 \nu_2 - \mu_2 \nu_1} , \quad (4.17a)$$

$$\frac{dy}{dt} = \frac{\mu_2 \omega_1 - \mu_1 \omega_2}{\mu_1 \nu_2 - \mu_2 \nu_1} \quad (4.17b)$$

where the parameters are determined by (4.13). In fact, every point in the wave pattern moves with this same speed. The entire wave pattern is stationary in a coordinate system that translates uniformly with a velocity given by (4.17). Every KP solution of genus 2 with  $\mu_1 \nu_2 \neq \mu_2 \nu_1$  is stationary in a coordinate system appropriate to that wave, just as every cnoidal wave is stationary in an appropriately translating, one-dimensional coordinate system.

For completeness, we now summarize a different description of these KP solutions of genus 2, in terms of the underlying algebraic geometry. More details are given by Dubrovin (1981). Every indecomposable  $2 \times 2$  Riemann matrix (real or not) corresponds to a compact Riemann surface of genus 2. This complex surface is a hyperelliptic curve, defined by an equation of the form

$$w^2 = P(Z; 5) \quad \text{or} \quad w^2 = P(Z; 6), \quad (4.18)$$

where  $P(Z; n)$  is a polynomial of degree  $n$ . The surface is topologically equivalent to a sphere with two handles. Given such a surface, there is a well-defined procedure to generate a Riemann matrix on the Riemann surface. Moreover, given a surface and a fixed reference point on it, there is a mapping (the Abel mapping) from each point on the Riemann surface to a point on a complex 2-torus, called the Jacobi variety. This torus is coordinatized by two independent complex variables. For example, if we allowed  $\phi_1$  and  $\phi_2$  in (4.1) to take on complex values, they would coordinatize the torus. Each of these complex variables has two complex periods, and at most one real period.

The inverse problem may be stated as follows. Given a Riemann matrix,  $B$ , did it come from a Riemann surface? If so, from which one? For genus 2, the solution as follows:  $B$  corresponds to a Riemann surface of genus 2 if and only if  $B$  is indecomposable. Here is how to construct the curve (4.18) explicitly, if  $B$  is indecomposable. Construct a theta function as in (4.3), and use (2.8) to scale  $\mu_1 = 1$ . Then (4.13c) has the form (4.18), and defines the hyperelliptic curve. Points on the Riemann surface are identified by the complex variable  $(\mu_4/\mu_1)$ . This surface has 6 branch points (called Weierstrass points), defined by  $P(Z; 6) = 0$ . These 6 points represent the 6 KdV solutions discussed above.

Every KP solution of genus 2 has eight free parameters:  $(b, d, \lambda)$  in (4.6), and  $(\mu_1, \mu_2, \nu_1, \phi_{10}, \phi_{20})$  in (4.1). Then  $(\nu_2, \omega_1, \omega_2)$  in (4.1) are determined by (4.13). Of the eight parameters,  $\phi_{10}$  and  $\phi_{20}$  affect neither the dynamics nor the algebraic geometry. Of the remaining six, we may always

choose

$$\mu_1 = 1 \quad \text{and} \quad \nu_1 = 0 ,$$

by using (2.7) and (2.8). Thus each KP solution of genus 2 has just four parameters that cannot be changed by a Lie point-symmetry of (1.8). Those are  $(b, d, \lambda)$ , which determine the underlying Riemann surface, and  $(\mu_2/\mu_1)$ , which determines the fixed reference point on that surface.

### 4.3 A soliton representation

By using twice the Poisson sum formula (3.7), one easily shows the equivalence of the following three representations:

$$\theta(\phi_1, \phi_2; B) = \sum_{m_2} \exp \left\{ \frac{d m_2^2}{2} + i m_2 \phi_4 \right\} * \sum_{m_1} \exp \left\{ \frac{1}{2} b (m_1 + \lambda m_2)^2 + i (m_1 + \lambda m_2) \phi_1 \right\}, \quad (4.7)$$

$$\theta(\phi_1, \phi_2; B) = \left( \frac{2\pi}{-b} \right)^{1/2} \sum_{n_1} \exp \left\{ \frac{2}{b} \left( \pi n_1 + \frac{\phi_1}{2} \right)^2 \right\} * \sum_{m_2} \exp \left\{ \frac{d m_2^2}{2} + i m_2 (\phi_4 - 2 \pi n_1 \lambda) \right\}, \quad (4.19a)$$

$$\theta(\phi_1, \phi_2; B) = \frac{2\pi}{(bd)^{1/2}} \sum_{n_1} \exp \left\{ \frac{2}{b} \left( \pi n_1 + \frac{\phi_1}{2} \right)^2 \right\} * \sum_{n_2} \exp \left\{ \frac{2}{d} \left( \pi (n_2 - \lambda n_1) + \frac{\phi_4}{2} \right)^2 \right\}, \quad (4.19b)$$

where all sums are taken over  $(-\infty, \infty)$ . It is numerically efficient to use (4.7) if both  $b$  and  $d$  are large (and negative), to use (4.19a) if  $d$  is large with  $b$  small, and to use (4.19b) if both  $b$  and  $d$  are small. (The other possibility,  $b$  large with  $d$  small, is excluded by (4.6).) Note from (4.19b) that if  $(b, d, \lambda, \phi_1, \phi_2)$  are all real, then  $\theta$  is the sum of positive terms. Therefore  $\theta$  never vanishes, and the associated KP solution is real and bounded. The corresponding formulae for the 4-component vector of theta constants are:



$$\hat{\theta}[\vec{p}] = \sum_{m_2} \exp \left\{ d (m_2 + p_2)^2 \right\} \cdot \sum_{m_1} \exp \left\{ b (m_1 + p_1 + \lambda (m_2 + p_2))^2 \right\} . \quad (4.10)$$

$$\hat{\theta}[\vec{p}] = \left( \frac{\pi}{-b} \right)^{1/2} \sum_{m_2=-\infty}^{\infty} \exp \left\{ d (m_2 + p_2)^2 \right\} \cdot \left[ 1 + 2 \sum_{n_1=1}^{\infty} \exp \left\{ \frac{(\pi n_1)^2}{b} \right\} \cdot \cos \left\{ 2\pi n_1 (p_1 + \lambda (m_2 + p_2)) \right\} \right], \quad (4.20a)$$

$$\hat{\theta}[\vec{p}] = \frac{\pi}{(bd)^{1/2}} \sum_{n_1} \exp \left\{ \frac{(\pi n_1)^2}{b} - 2\pi i n_1 p_1 \right\} \cdot \sum_{n_2} \exp \left\{ \pi^2 \frac{(n_2 - \lambda n_1)^2}{d} - 2\pi i n_2 p_2 \right\} \quad (4.20b)$$

We now propose a soliton interpretation of the KP solutions of genus 2.

a) These solutions can be viewed as a doubly periodic array of interacting solitons. This description becomes exact in the limit  $b \rightarrow 0$ ,  $d \rightarrow 0$ .

b) Away from the regions of interaction, the crests of the individual solitons lie along  $\phi_3 = \text{const}$  and  $\phi_4 = \text{const}$ , as defined in (4.11). (Within a region of interaction, of course, individual solitons cannot even be defined unambiguously.) Thus the edges of the period parallelogram lie along  $\phi_1 = \text{const}$ . and  $\phi_2 = \text{const}$ ., but the wave crests ordinarily do not. According to (4.11), therefore,  $\lambda$  and  $\bar{\lambda}$  determine the angles of rotation of the wave crests from the directions of periodicity. This is a direct geometric-dynamical interpretation of the off-diagonal term in the Riemann matrix: it determines how strongly the waves of the two families ( $\phi_3$  and  $\phi_4$ ) affect each other.

c) A given wave crest experiences a phase shift from each of its interactions with waves of the other family. The strength of the phase shift is proportional by  $\lambda$  (or  $\bar{\lambda}$ ).

The soliton interpretation can be proved in the limit  $b \rightarrow 0$ ,  $d \rightarrow 0$ , so we consider that case first. Note that (4.19b) can be written as

$$\theta = \frac{2\pi}{(bd)^{1/2}} \sum_{n_1} \exp \left\{ \frac{2}{b} \left( \pi n_1 + \frac{\phi_1}{2} \right)^2 \right\} \cdot \sum_{n_2} \exp \left\{ \frac{2}{d} \left( \pi n_2 + \frac{\phi_2}{2} - \lambda \left( \pi n_1 + \frac{\phi_1}{2} \right) \right)^2 \right\} \quad (4.21)$$

In the limit of  $b \rightarrow 0$ ,  $d \rightarrow 0$  and in a neighborhood of  $(\phi_1 = \pi, \phi_2 = \pi)$ , the four terms in (4.21) corresponding to  $(n_1 = -1, 0)$ ,  $(n_2 = -1, 0)$  dominate. Retaining only those terms in (4.21) yields

$$\begin{aligned} \theta = \frac{\pi}{(bd)^{1/2}} & \left[ \exp \left\{ \left( \frac{2}{b} + \frac{2\lambda^2}{d} \right) \left( \frac{\phi_1}{2} - \pi \right)^2 + \frac{2}{d} \left( \frac{\phi_2}{2} - \pi \right)^2 - \frac{4\lambda}{d} \left( \frac{\phi_1}{2} - \pi \right) \left( \frac{\phi_2}{2} - \pi \right) \right\} + \right. \\ & \exp \left\{ \left( \frac{2}{b} + \frac{2\lambda^2}{d} \right) \left( \frac{\phi_1}{2} - \pi \right)^2 + \frac{2}{d} \left( \frac{\phi_2}{2} \right)^2 - \frac{4\lambda}{d} \left( \frac{\phi_1}{2} - \pi \right) \left( \frac{\phi_2}{2} \right) \right\} + \\ & \exp \left\{ \left( \frac{2}{b} + \frac{2\lambda^2}{d} \right) \left( \frac{\phi_1}{2} \right)^2 + \frac{2}{d} \left( \frac{\phi_2}{2} - \pi \right)^2 - \frac{4\lambda}{d} \left( \frac{\phi_1}{2} \right) \left( \frac{\phi_2}{2} - \pi \right) \right\} + \\ & \left. \exp \left\{ \left( \frac{2}{b} + \frac{2\lambda^2}{d} \right) \left( \frac{\phi_1}{2} \right)^2 + \frac{2}{d} \left( \frac{\phi_2}{2} \right)^2 - \frac{4\lambda}{d} \left( \frac{\phi_1}{2} \right) \left( \frac{\phi_2}{2} \right) \right\} \right] + \dots \quad (4.22) \end{aligned}$$

Substituting this into (4.3) yields the approximate KP solution

$$f(x, y, t) = 2\partial_x^2 \ln \left[ 1 + \exp \eta_3 + \exp \eta_4 + \exp(\eta_3 + \eta_4 + A) \right] + \frac{2\mu_1^2}{b} + \frac{2(\mu_2 - \lambda\mu_1)^2}{d}, \quad (4.23)$$

where

$$\begin{aligned} \eta_4 &= -\frac{2\pi}{d} (\phi_2 - \lambda \phi_1 - \pi) = -\frac{2\pi}{d} (\phi_4 - \pi), \\ \eta_3 &= 2\pi \left( \frac{1}{b} + \frac{\lambda^2}{d} \right) (\phi_1 - \pi) + 2\lambda\pi \phi_2/d = -2\pi \left( \frac{1}{b} + \frac{\lambda^2}{d} \right) (\phi_3 - \pi), \\ A &= -\frac{4\pi^2\lambda}{d}, \end{aligned}$$

and we have used (4.6) and (4.11). The parameters in (4.23) are constrained by (4.13), which in this limit becomes

$$\begin{aligned} c_3 &= 3\rho_3^2 + \left( 2\pi\mu_3 \left( \frac{1}{b} + \frac{\lambda^2}{d} \right) \right)^2, \\ c_4 &= 3\rho_4^2 + \left( \frac{2\pi\mu_4}{-d} \right)^2, \end{aligned} \quad (4.24)$$

$$\exp\left(\frac{4\pi^2\lambda}{-d}\right) = \frac{\left[ \left( \frac{2\pi\mu_4}{d} \right) - \left( \frac{2\pi\mu_3(b\lambda^2 + d)}{bd} \right) \right]^2 - (\rho_4 - \rho_3)^2}{\left[ \left( \frac{2\pi\mu_4}{-d} \right) + \left( \frac{2\pi\mu_3(b\lambda^2 + d)}{bd} \right) \right]^2 - (\rho_4 - \rho_3)^2}.$$

Note that the wave speeds remain finite in this limit only if  $(\mu_4/d)$  remains finite as  $d \rightarrow 0$ , and if  $(\mu_1/b)$  remains finite as  $b \rightarrow 0$ . But (4.23) and (4.24) represent an exact solution of (1.8)! They describe two solitons, interacting obliquely on a depressed mean level (cf. Satsuma (1976) or Ablowitz & Segur 1981, p. 189). Away from the region of interaction, the crests of the individual solitons lie along  $\phi_3 = \text{constant}$  and  $\phi_4 = \text{constant}$ . The phase shift of the  $\phi_4$  - soliton due to the interaction is obtained from (4.23) by letting  $\phi_3 \rightarrow \pm \pi$ , holding  $\phi_4$  fixed. The result is

$$\Delta \phi_4 = 2\pi\lambda, \quad \Delta \phi_3 = 2\pi\tilde{\lambda}. \quad (4.25)$$

The largest neighborhood of  $(\phi_1 = \pi, \phi_2 = \pi)$  in which the four terms in (4.22) dominate is precisely a period parallelogram centered at  $(\phi_1 = \pi, \phi_2 = \pi)$ . Moreover, in the corresponding period parallelogram centered at  $(\phi_1 = (2N_1 + 1)\pi, \phi_2 = (2N_2 + 1)\pi)$ , there are in (4.21) exactly four dominant terms, indexed by  $(n_1 + N_1 = -1, 0)$ ,  $(n_2 + N_2 = -1, 0)$ . Thus the entire series in (4.21) represents a bi-periodic repetition of a 2-soliton solution of (1.8).

To summarize, we have now established the validity of the soliton-interpretation given above in the limit  $b \rightarrow 0$ ,  $d \rightarrow 0$ . Figure 2a shows a two-soliton solution of (1.8). Figure 2b shows a genus 2 solution with  $(b = -1, b\lambda^2 + d = -1)$ . The sense in which the solution of genus 2 represents a bi-periodic extension of the two soliton solution is evident.

What happens if  $b$  and  $d$  are not small? Given any  $b, d$  satisfying (4.6), then in a period parallelogram centered at  $(\phi_1 = \pi, \phi_2 = \pi)$ , the four terms in (4.22) are bigger than any of the others in (4.21), but they are not necessarily dominant in an asymptotic sense. Even so, there is a practical sense in which the wave crests still lie along  $\phi_3 = \text{constant}$  and  $\phi_4 = \text{constant}$ , provided this statement is interpreted correctly. This assertion is corroborated in Figure 3, which shows some typical KP solutions of genus 2 outside the soliton limit. Each wave pattern consists of a bi-periodic array of peaks, connected to each other in two directions by

ridges. The peaks are the interaction regions, where the two wave amplitudes "add". If those peaks were well separated as in Figure 2, the ridges would become the wave crests of the individual solitons. In Figure 3 the peaks are not well separated, and the "individual wave crests" are simply the lines of steepest descent and ascent across the saddles. We have sketched lines of constant  $\phi_3$  and constant  $\phi_4$  in Figures 2c, 3d, and 3g to show that they identify the wave crests (i.e., the lines of steepest ascent in the saddles) in the general case.

Do the wave crests still lie along  $\phi_3 = \text{constant}$  and  $\phi_4 = \text{constant}$ , even if  $b, d \rightarrow -\infty$ ?

(i) Suppose  $d \rightarrow -\infty$ , with  $b, \lambda$  finite. Then  $\bar{\lambda} \rightarrow 0$  from (4.11), so  $\phi_3 \rightarrow \phi_1$ . In this limit, therefore, the assertion is that the crests lie along  $\phi_1 = \text{constant}$  and  $\phi_4 = \text{constant}$ . Its validity for  $d \rightarrow -\infty$  follows easily from (4.19a); it is shown in Figure 1, which is actually a wave of genus 2 with  $b = -3$ ,  $d = -12$ ,  $\lambda = 0.3$ ,  $\mu_1 = \mu_2 = 0.5$ .

(ii) Suppose  $b \rightarrow -\infty$  and  $d \rightarrow -\infty$ . Then the appropriate representation of  $\theta$  is (4.7). We will show in §4.4 that the limit  $b \rightarrow -\infty$ ,  $d \rightarrow -\infty$  is nondegenerate only if  $\lambda \rightarrow 0$  as well. In this limit, therefore,  $\lambda \rightarrow 0$ ,  $\bar{\lambda} \rightarrow 0$  and the wave crests lie along the directions of periodicity.

In summary, we assert that the wave crests always lie along  $\phi_3 = \text{constant}$  and  $\phi_4 = \text{constant}$ . We have proved this assertion in certain limiting cases, and have offered supporting evidence in other cases.

Now we turn to point (c) above. We have already shown that in the limit  $b \rightarrow 0$ ,  $d \rightarrow 0$ , the phase shifts in each period of the lines  $\phi_3 = \text{constant}$  and  $\phi_4 = \text{constant}$  are given by (4.25). In fact, the validity of (4.25) does not rely on any special limits; it is a consequence of elementary geometry. In Appendix A, we prove the following theorem. Let  $\phi_1$  and  $\phi_2$  be independent real variables. For fixed  $\lambda$  and  $\bar{\lambda}$  ( $0 \leq \bar{\lambda}^2 \leq \lambda^2 \leq 1/4$ ), define  $\phi_3$  and  $\phi_4$  by (4.11) on  $0 \leq \phi_1 < 2\pi$ ,  $0 \leq \phi_2 < 2\pi$ , and by periodic extension outside this rectangle. Then once each period,  $\phi_3 = \text{constant}$  and  $\phi_4 = \text{constant}$  experience discontinuous shifts whose size and direction are given by (4.25).

#### 4.4 Limits and asymptotic formulae

The KP solutions of genus 2 generalize and unify several simple models of waves in shallow waters, including one-dimensional cnoidal waves, obliquely superposed infinitesimal waves (i.e., Fourier modes), and 2-soliton formulae. We show next how these previously known results emerge from the KP solutions of genus 2 in various limits. We also develop asymptotic formulae that will be useful in §5.

(a) Limit of one weak wave:  $d \rightarrow -\infty$

If  $d \rightarrow -\infty$  in (4.7), then

$$\begin{aligned} \theta &= \sum_m \exp \left\{ \frac{1}{2} b m^2 + i m \phi_1 \right\} + \\ &+ e^{d/2} \left[ \exp(i \phi_4) \sum_m \exp \left\{ \frac{1}{2} b (m + \lambda)^2 + i (m + \lambda) \phi_1 \right\} + (*) \right] \\ &+ O(e^{2d}), \end{aligned} \quad (4.27)$$

where (\*) denotes complex conjugate. Comparing this with (3.2) shows that at leading order, (4.27) represents a one-dimensional cnoidal wave with phase variable  $\phi_1$ . There is no phase shift of the cnoidal wave at this order because the second wave is too weak. The next order describes a weak perturbation of the cnoidal wave, with an amplitude  $O(e^{d/2})$ . This perturbation solves the equation obtained by linearizing (1.8) about a cnoidal wave. We will refer to these weak perturbations as "linear waves". An example of such a perturbed cnoidal wave is shown in Figure 3a.

The limiting form of (4.12) depends on whether or not  $b$  remains finite as  $d \rightarrow -\infty$ . Suppose  $b$  remains finite. Then for either  $p_1 = 0$  or  $p_1 = 1/2$  in (4.10),

$$\hat{\theta}_{[0^1]}^{p_1} = \sum_m \exp \left\{ b (m + p_1)^2 \right\} + e^d \sum_m \left[ \exp \left\{ b (m + p_1 + \lambda)^2 \right\} + \exp \left\{ b (m + p_1 - \lambda)^2 \right\} \right] + o(e^{4d}),$$

$$\hat{\theta}_{[1/2]}^{p_1} = e^{d/4} \left[ \sum_m \exp \left\{ b (m + p_1 + \lambda/2)^2 \right\} + \exp \left\{ b (m + p_1 - \lambda/2)^2 \right\} + o(e^{2d}) \right].$$

When these limiting forms are substitute into (4.12), the two equations with  $p_2 = 0$  become

$$D \hat{\theta}_1 \begin{bmatrix} p_1 \\ 0 \end{bmatrix} - (\mu_1 \omega_1 + 3 \nu_1^2) \partial_b \hat{\theta}_1 \begin{bmatrix} p_1 \\ 0 \end{bmatrix} + 4\mu_1^4 \partial_b^2 \hat{\theta}_1 \begin{bmatrix} p_1 \\ 0 \end{bmatrix} = 0, \quad (4.28)$$

where  $\hat{\theta}_1 \begin{bmatrix} p_1 \\ 0 \end{bmatrix} = \sum_m \exp \left\{ b (m + p_1)^2 \right\}.$

These are exactly the equations one obtains for a cnoidal wave. If  $D$  is eliminated from (4.28), the resulting equation is equivalent to (3.10). The other two equations in (4.12), with  $p_2 = 1/2$ , determine the speed and orientation of the linear perturbation.

(b) The small amplitude limit:  $d \rightarrow -\infty$ ,  $b \rightarrow -\infty$ ,  $\mu_1 = 0$  (1)

In this limit, (4.7) becomes

$$\theta = 1 + 2 \exp \left\{ \frac{b}{2} \right\} \cos \phi_1 + 2 \exp \left\{ \frac{1}{2} (b\lambda^2 + d) \right\} \cos \phi_2,$$

corresponding to an approximate KP solution of

$$f(x, y, t) = 2\mu_1^2 \exp\left\{\frac{b}{2}\right\} \cos \phi_1 + 2\mu_2^2 \exp\left\{\frac{1}{2}(b\lambda^2 + d)\right\} \cos \phi_2, \quad (4.29)$$

where  $\phi_1$  and  $\phi_2$  are defined by (4.1). Clearly (4.29) represents a linear superposition of two Fourier modes of small amplitude, and one expects (4.13) to produce the dispersion relation of the linearized problem in this limit. This expectation is correct. One shows that in this limit, (4.13) becomes

$$\mu_1 \omega_1 + 3 v_1^2 = \mu_1^4, \quad (4.30a)$$

$$\mu_2 \omega_2 + 3 v_2^2 = \mu_2^4, \quad (4.30b)$$

$$\exp(b\lambda) = \frac{(\mu_1 - \mu_2)^2 + (\rho_1 - \rho_2)^2}{(\mu_1 + \mu_2)^2 + (\rho_1 - \rho_2)^2}, \quad (4.30c)$$

where we have used (4.1b). Because  $b \rightarrow -\infty$  with  $\mu_1, \mu_2$  finite, (4.30c) can be satisfied only if  $\lambda \rightarrow 0$  as well. Then one may prescribe two Fourier modes in (4.29) arbitrarily, and (4.30c) determines  $(b\lambda)$ . Because  $\lambda \rightarrow 0$ , the wave crests and the lines of periodicity coalesce in this limit.

For use in §5, we also give the next corrections to (4.30a,b). As  $b \rightarrow -\infty, d \rightarrow -\infty$ , with  $(b\lambda)$  finite,

$$\begin{aligned} -\psi_1 &= \frac{c_1 - 3\rho_1^2}{\mu_1^2} \sim -1 + 24 \exp(b) - 12 \exp(b\lambda^2 + d) \cdot M/\mu_1^6, \\ -\psi_2 &= \frac{c_2 - 3\rho_2^2}{\mu_2^2} \sim -1 + 24 \exp(b\lambda^2 + d) - 12 \exp(b) \cdot M/\mu_2^6, \end{aligned} \quad (4.31)$$



where

$$M = \mu_1^2 \mu_2^2 \left[ \left( \mu_1 + \mu_2 \right)^2 \exp(b\lambda) + \left( \mu_1 - \mu_2 \right)^2 \exp(-b\lambda) - 2 \left( \mu_1^2 + \mu_2^2 \right) \right].$$

(c) Limit of one soliton plus perturbation:  $d \rightarrow -\infty$ ,  $b \rightarrow 0$

Consider (4.19a) in the limit  $b \rightarrow 0$ ,  $d \rightarrow -\infty$ , and in a neighborhood of  $\phi_1 = \pi$ ;

$$\theta = \left( \frac{2\pi}{-b} \right)^{1/2} \exp \left\{ \frac{\phi_1^2}{2b} \right\} \left[ 1 + \exp \left\{ \frac{2\pi}{b} (\pi - \phi_1) \right\} + \right. \\ \left. + 2 \exp \left( \frac{d}{2} \right) \left( \cos \phi_4 + \exp \left\{ \frac{2\pi}{b} (\pi - \phi_1) \right\} \cos (\phi_4 + 2\pi\lambda) \right) \right] \quad (4.32)$$

where  $\phi_1$  and  $\phi_4$  are defined in (4.1), (4.11). The parameters of this solution satisfy

$$\left( \frac{2\pi\mu_1}{-b} \right) \left( \frac{2\pi\omega_1}{-b} \right) + 3 \left( \frac{2\pi\nu_1}{-b} \right)^2 + \left( \frac{2\pi\mu_1}{-b} \right)^4 \left[ 1 + \frac{3b}{\pi^2} \right] = 0 \quad (4.33a)$$

$$\mu_4 \omega_4 + 3\nu_4^2 - \mu_4^4 = 0 \quad (4.33b)$$

$$\left[ \left( \frac{2\pi\mu_1}{-b} \right) \nu_4 - \left( \frac{2\pi\nu_1}{-b} \right) \mu_4 \right]^2 =$$

$$\left( \frac{2\pi\mu_1}{-b} \right)^2 \mu_4^2 \left[ \left( \frac{2\pi\mu_1}{-b} \right)^2 + 2 \mu_4 \left( \frac{2\pi\mu_1}{-b} \right) \cot(\pi\lambda) - \mu_4^2 \right]. \quad (4.33c)$$

At leading order in  $\exp(d/2)$ , (4.32) gives a one-dimensional soliton as  $b \rightarrow 0$ . Equation (4.33a) shows that the speed of this soliton remains finite as

$b \rightarrow 0$  only if  $\mu_1 \rightarrow 0$  so that  $(\mu_1/b)$  remains finite. At second order, (4.33b) shows that the speed of the linear wave is unaffected by the presence of the soliton of leading order. The next order corrections to (4.33a,b) are

$$-\psi_1 = \frac{c_1 - 3\rho_1^2}{\mu_1^2} - \left(\frac{2\pi}{-b}\right)^2 \left[1 + \frac{3b}{\pi^2}\right] + 24 \exp(d) \left[ \left(1 - \cos(2\pi\lambda)\right) (1 - z^2) z^2 + 2 \sin(2\pi\lambda) \cdot z^3 \right], \quad (4.34)$$

$$-\psi_4 = \frac{c_4 - 3\rho_4^2}{\mu_1^2} - 1 + 24 \exp(d) + \left(\frac{3b}{\pi^2}\right) z^{-2},$$

where  $z = \frac{-b\mu_4}{2\pi\mu_1}$ .

(d) Limit of two solitons:  $b \rightarrow 0$ ,  $d \rightarrow 0$

This limit was already discussed in § 4.3, and from quite a different viewpoint by McKean (1979). The higher order corrections to the wave speeds are

$$-\psi_3 = \frac{c_3 - 3\rho_3^2}{\mu_3^2} - \left(\frac{2\pi(b\lambda^2 + d)}{bd}\right)^2 + \frac{12}{b} \left(\frac{\mu_1}{\mu_3}\right)^2 + \frac{12}{d} \left(\frac{\mu_4}{\mu_3}\right)^2, \quad (4.35)$$

$$-\psi_4 = \frac{c_4 - 3\rho_4^2}{\mu_4^2} - \left(\frac{2\pi}{-d}\right)^2 \left(1 + \frac{3d}{\pi^2}\right) + \frac{12}{b} \left(\frac{\mu_1}{\mu_4}\right)^2.$$

Note that both  $\psi_3$  and  $\psi_4$  are singular in this limit, although the wave speeds are not.

## 5. Using wave measurements to calibrate the model

The KP solutions of genus 2 have eight free parameters:  $b$ ,  $\lambda$ ,  $d$  in (4.6), and  $\mu_1$ ,  $\mu_2$ ,  $v_1$ ,  $\phi_{10}$ ,  $\phi_{20}$  in (4.1). In order to use these solutions as models of two-dimensional periodic waves in shallow water, one must be able to infer these parameters from measurements of the water wave. The purpose of this section is to show how to infer these parameters from two different kinds of wave measurements. At this time, we have not proven rigorously the validity of every step of our algorithms, but we have identified the major unproven points in five conjectures.

Certainly the KP solutions of genus 2 cannot describe all possible waves in shallow water. Every two-dimensional KP solution of genus 2 that is generated by a real Riemann matrix, as discussed in §4, necessarily has the following four properties, so the water waves that they model should also have these properties.

- (i) The solution is real-valued and bounded, for all real  $(x, y, t)$ .
- (ii) It is spatially periodic in two independent directions, which we may identify with  $\phi_1$  and  $\phi_2$  in (4.1). If  $\phi_1$  and  $\phi_2$  are not collinear, then  $\mu_1 v_2 \neq \mu_2 v_1$ . We call a solution two-dimensional if  $\mu_1 v_2 \neq \mu_2 v_1$ .
- (iii) There is a uniformly translating coordinate system in which the entire two-dimensional wave pattern is stationary.
- (iv) The common features of two-dimensional KP solutions of genus 2 can be inferred from the examples shown in Figures 2 and 3. Within each period parallelogram, there is a single wave peak. It occurs in a region of interaction of two intersecting "wave crests". The crests connect adjacent peaks, and may appear as ridges of constant wave amplitude, as in Figure 2, or merely as lines of steepest descent and ascent between neighboring peaks, as in Figure 3. Ordinarily, each wave crest experiences a phase shift in crossing a region of interaction.

CONJECTURE A: Out of all possible KP solutions, the only ones that are (i) real-valued, (ii) genuinely two-dimensional, and (iii) stationary in some uniformly translating coordinate system are those in the form (1.9), with genus 2.

The first step in comparing water wave data to KP solutions is to transform the physical data into KP data in the form  $f(x,y,t)$ . This is accomplished via (2.4), with  $F = 0$  in (2.4b), once we have decided on: (i) the magnitude of  $\epsilon$ ; and (ii) the direction of the  $x$ -axis in the horizontal plane. A reasonable definition of  $\epsilon$  is given below (2.3). A more refined definition is unnecessary, because adjustments of  $\epsilon$  can be absorbed into  $\beta$  in (2.8).

The direction of the  $x$ -axis should be approximately the principal direction of wave propagation. Small readjustments of this direction can be absorbed into  $\alpha$  in (2.7), but one can expect significant errors if (2.5) is used to model water waves propagating in a direction much different from the  $x$ -direction. In the remainder of this section, we assume that these choices already have been made, and that the water wave data already have been transformed via (2.4) into KP data.

#### 5.1 Wavelength and velocity data

As discussed in §3, a cnoidal wave is determined to within a translation by specifying its wavelength and speed of propagation. The KP solutions of genus 2 that are of interest here are two-dimensional generalizations of cnoidal waves, so one expects that a similar procedure might succeed here as well.

Here is a precise statement of the mathematical problem. Let  $f(x,y,t)$  represent a particular two-dimensional KP solution of genus 2. Then  $f$  has the form (4.8), and is specified completely by the eight free parameters in (4.1) and (4.6). Identify eight measurements of  $f(x,y,t)$  which are sufficient to determine these parameters, and give an explicit algorithm to find the parameters from the measurements.

The practical problem differs from this in two respects.

(i) Instead of  $f(x,y,t)$ , a two-dimensional wave in shallow water is observed. The wave pattern has the four properties listed above that are common to all two-dimensional KP solutions of genus 2, and we use Conjecture A to assert that the water wave can be represented by a two-dimensional KP solution of genus 2.

(ii) We do not necessarily require that the eight free parameters be determined from only eight measurements. In practice it is often convenient to take one or two extra measurements and to minimize the effect of measurement errors.

In any case, how does one find experimentally the free parameters of a two-dimensional KP solution of genus 2? The method proposed here has eight fundamental steps.

(1) Find a period parallelogram at a fixed time,  $\bar{t}$ .

This step is based on a procedure due to Arnol'd (1978, p.276). Start with any point,  $(\bar{x}, \bar{y}, \bar{t})$ . (From here to step(5),  $t$  remains fixed at  $\bar{t}$ , and we shall suppress it.) Because  $f(x,y)$  is a two-dimensional, bi-periodic function, there is an infinite array of points in the plane at which  $f$  is periodic to  $f(\bar{x}, \bar{y})$ ; i.e., at which  $f$  and all of its derivatives match those at  $(\bar{x}, \bar{y})$ . This collection of points is called a period lattice (relative to  $(\bar{x}, \bar{y})$ ). Pick any point in this period lattice other than  $(\bar{x}, \bar{y})$ . Denote the line through these two points by  $\bar{\phi}_1 = \text{const.}$  Out of all points on this line that are in the period lattice, denote one closest (but not equal) to  $(\bar{x}, \bar{y})$  by  $(x_1, y_1)$ .

Out of all points off of this line that are in the period lattice, denote one closest to the line by  $(x_2, y_2)$ . The two line segments connecting  $(\bar{x}, \bar{y})$  to  $(x_1, y_1)$  and  $(\bar{x}, \bar{y})$  to  $(x_2, y_2)$ , are two sides of a parallelogram with the following properties.

(i) The parallelogram contains exactly four points of the period lattice. These four points lie at its vertices. (For a proof, see Arnol'd, 1978).

(ii) The parallelogram is a period parallelogram of  $f$ ; i.e.,  $f(x,y)$  may be

defined in the entire plane by periodically extending the function defined on the parallelogram.

(iii) The parallelogram cannot be replaced by any smaller parallelogram without losing information about  $f(x,y)$ .

(iv) The enclosed area of this parallelogram can be measured directly, and it provides information about the parameters in question, through (4.16). The method described here does not use this area explicitly, but it can be used as a check on the results obtained.

Define  $\bar{\phi}_1(x,y)$  as in (4.1), and choose  $(\bar{\mu}_1, \bar{\nu}_1)$  so that: (i)  $\bar{\phi}_1 = 0$  along one side of the period parallelogram; (ii)  $\bar{\phi}_1 = 2\pi$  along the opposite side. Define  $\bar{\phi}_2(x,y)$  in the same way using the other two sides of the parallelogram.

(2) Find the wave crests

Within each period parallelogram,  $f(x,y)$  has one complete region of interaction, containing the maximum of  $f(x,y)$ . As demonstrated in § 4, each region of interaction is connected in two directions (related to  $\phi_3$  and  $\phi_4$  in (4.11)) by saddles; see especially Figures 3c, 3d, 3f, 3g. We now define a "wave crest" to be a straight line segment along a saddle connecting two adjacent regions of interaction. These are wave crests in the sense that along any other line that intersects one of these line segments transversely,  $f(x,y)$  attains a local maximum at the wave crest. In this paper, "wave crest" always refers to a saddle between two peaks, and not the peak itself.

Emanating from every region of interaction are two pairs of parallel wave crests. We will eventually associate the pair possessing the larger wave amplitudes at their saddles with  $\phi_3$ , and the pair possessing the smaller wave amplitudes with  $\phi_4$ .

CONJECTURE B: The wave crests in each period parallelogram are defined by  $\phi_3 = \text{const.}$  and  $\phi_4 = \text{const.}$ , where  $\phi_3$  and  $\phi_4$  are defined by (4.11).

(3) Find a basic period parallelogram

The period parallelogram found in step (1) is not unique. Figure 5 shows two dissimilar period parallelograms on the same period lattice, and enclosing the same area. Denote the sides of the two parallelograms  $\bar{\phi}_1, \bar{\phi}_2 = \text{const}$ , and by  $\hat{\phi}_1, \hat{\phi}_2 = \text{const}$ , respectively, as shown in Figure 5. The two sets of coordinates in Figure 5 are related by

$$\begin{pmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{\phi}_1 \\ \bar{\phi}_2 \end{pmatrix}.$$

as the reader may verify.

(5.1)

More generally, let  $(\bar{\phi}_1, \bar{\phi}_2) = \text{const}$  denote the sides of any other period parallelogram on this same period lattice, that encloses the same area as that of  $(\hat{\phi}_1, \hat{\phi}_2)$ . These coordinates are related by

$$\begin{pmatrix} \bar{\phi}_1 \\ \bar{\phi}_2 \end{pmatrix} = M \begin{pmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{pmatrix}$$

(5.2)

where  $M$  is an integer,  $2 \times 2$ , unimodular matrix (i.e., integer elements and determinant  $= \pm 1$ ). These are precisely the transformations used to reduce a real Riemann matrix to its basic form, because the symplectic transformations among equivalent Riemann matrices correspond precisely to the transformations among period parallelograms with the same enclosed area on a fixed period lattice.

The two kinds of transformations relate to theta functions as follows. A real  $2 \times 2$  Riemann matrix, plus a particular period parallelogram on a lattice, determines a theta function, as in (4.3). Suppose the Riemann matrix is transformed to an equivalent one by a symplectic transformation. To keep the same theta function, one must also transform the period parallelogram by the corresponding integer unimodular matrix, as in (5.2). Thus, for a given theta function, each equivalent Riemann matrix corresponds to its own period parallelogram on a fixed period lattice. (Here and in what follows "period parallelogram" refers to both the single four-sided figure and to the tiling of the entire plane by this figure.)

Recall that a real, indecomposable,  $2 \times 2$  Riemann matrix is equivalent to a basic Riemann matrix, for which

$$0 < \bar{\lambda}^2 \leq \lambda^2 \leq \frac{1}{4} . \quad (5.3)$$

Corresponding to the basic Riemann matrix is a particular period parallelogram, which we also call basic. Here is how to identify it. Define  $\phi_1(x,y)$  and  $\phi_2(x,y)$  as in (4.1), so that  $\phi_1(x,y)/2\pi$  takes on sequential integer values on two parallel sides of the parallelogram, and  $\phi_2(x,y)/2\pi$  takes on sequential integer values on the other two sides. The basic period parallelogram is identified by two properties.

(i) It is related to the parallelogram in step (1) by an integer unimodular transformation. In particular, it encloses the same area as the parallelogram in step (1).

(ii) The wave crests are related to the lines of periodicity of the basic parallelogram by

$$\phi_3(x,y) = \phi_1(x,y) - \bar{\lambda} \phi_2(x,y) , \quad (5.4)$$

$$\phi_4(x,y) = \phi_2(x,y) - \lambda \phi_1(x,y) ,$$

where  $(\lambda, \bar{\lambda})$  satisfy (5.3) and

$$\lambda \bar{\lambda} > 0 . \quad (5.5)$$

We show in Appendix B that the basic period parallelogram is uniquely defined by these two properties. Here are some other properties, which follow from these and from the results in §4.

(iii) If  $\phi_1$  is increased by  $2\pi$ , holding  $\phi_2$  fixed,  $\phi_3$  changes by  $2\pi$ . Conversely, if  $\phi_2$  is increased by  $2\pi$ , holding  $\phi_1$  fixed,  $\phi_3$  changes by  $(-2\pi \bar{\lambda})$ .

(iv) The wave crests ( $\phi_3$ ) associated with the larger amplitude waves



experience a smaller phase shift ( $2\pi\bar{\lambda}$ ), while the smaller amplitude waves feel a larger phase shift ( $2\pi\lambda$ ) as a result of their interaction.

(v) Given a period parallelogram, the wave crests emanating from a vertex are constrained by (5.3) to lie in certain two-dimensional cones.

(vi) If  $\phi_3$  lies to the left of  $\phi_1$ , then  $\phi_4$  lies to the right of  $\phi_2$ , because of (5.5). The interaction does not represent a common rotation of both wave crests.

#### (4) Measure the wave numbers

At this point, we have identified the basic period parallelogram, and have located the two sets of wave crests within the parallelogram. Now we may measure the wave numbers ( $\mu_1, \mu_2, \nu_1, \nu_2$ ) directly. We also measure  $\lambda$  directly.

(i)  $\phi_3 = \text{const}$  on the wave crests with the larger amplitude waves, and with the smaller deviation from the nearest line of periodicity.

(ii)  $\phi_1 = \text{const}$  on the two sides of the parallelogram most nearly aligned with  $\phi_3 = \text{const}$ .

(iii) The straight lines along these two sides are defined by

$$\mu_1 x + \nu_1 y + \bar{\phi}_{10} = 0, \quad \mu_1 x + \nu_1 y + \bar{\phi}_{10} = 2\pi, \quad (5.6)$$

so that  $(\mu_1, \nu_1)$  can be found by direct measurement.

(iv) From the other two sides of the parallelogram, one also finds  $(\mu_2, \nu_2)$  by direct measurement. The area of the parallelogram, of course, then satisfies (4.16).

(v)  $\lambda$  can be obtained either by measuring an angle and using (5.4), or by measuring the phase shift of the  $\phi_4$ -crests and using (4.25).

(vi) One obtains  $\bar{\lambda}$  in a similar way. From (4.11b),

$$d/b = \lambda/\bar{\lambda} - \lambda^2. \quad (5.7)$$

We use this information simply as a check on  $(d/b)$ , but it is not necessary to do so.

(5) Measure the wave speeds

Because the wave pattern is a two-dimensional KP solution of genus 2, it is stationary in a uniformly translating coordinate system. Measure this speed of translation. Then (4.17) determines  $(\omega_1, \omega_2)$ .

(6) Deduce the Riemann matrix

Given any basic Riemann matrix  $(b, d, \lambda)$ , and any real  $(\mu_4/\mu_1)$  satisfying (4.13d), then (4.13a, b) determine  $\Psi_1, \Psi_4$ . Suppose  $(\Psi_1, \Psi_4)$  were generated in this way from some  $(b, d, \lambda, (\mu_4/\mu_1))$ .

CONJECTURE C: Given  $(\lambda, (\mu_4/\mu_1), \Psi_1, \Psi_4)$ , (4.13a, b) can be inverted to find  $(b, d)$  uniquely.

We have not proven this assertion, but we offer the following evidence in support of it.

- (i) The validity of the corresponding statement for genus 1 can be proven.
- (ii) We have tested this conjecture numerically by first generating  $(\Psi_1, \Psi_4)$  from a particular Riemann matrix, then using (4.13a, b) in a simple Newton-type root-finder to solve for  $(b, d)$ , given  $(\Psi_1, \Psi_4, \lambda, \mu_4/\mu_1)$ . Our experience has been that the algorithm always converges quickly to the correct values, even if our starting values were chosen poorly.

Returning now to our two-dimensional KP solution of genus 2, in effect we have already measured  $(\Psi_1, \Psi_4, \lambda, (\mu_4/\mu_1))$ . Then it follows from Conjecture C that  $(b, d)$  are determined from (4.13a, b). Reasonably accurate starting values can be obtained from (4.31), (4.34) and (4.35).

Conjecture C is closely related to another conjecture that we never use explicitly, but which is implicit in this entire algorithm.

CONJECTURE D: Every bounded real-valued KP solution of the form (1.9) is generated by a real-valued Riemann matrix.

(7) Check Consistency

(i) The ratio  $(b/d)$  has been determined independently in steps (4) and (6). These must agree.

(ii) Both sides of (4.13c) have been determined independently. These must also agree.

(8) Measure the phase constants

If desired,  $\phi_{10}$ ,  $\phi_{20}$  in (4.1) can be measured for a particular coordinate system. These have no dynamic significance. In any case, this completes the algorithm.

## 5.2 Initial Data

A second way to specify these waves is with initial data: to specify the elevation of the water surface everywhere at some fixed time (which we call  $t = 0$ ). Suppose  $f(x,y,0)$  is genuinely two-dimensional, is periodic in two independent horizontal directions, and is given completely in one period parallelogram. We must answer two questions: (i) Is this the initial data of a KP solution of genus 2? (ii) If so, how does the wave evolve in time? This method of solution was first given by Segur, Finkel & Philander (1983).

Is this the initial data of a two-dimensional KP solution of genus 2? Conjecture A is not effective for initial data, because we have no information about the time evolution of the solution. Instead, we assume that  $f(x,y,0)$  has the form (4.8) and derive necessary conditions that  $f$  must satisfy in order to be initial data of a two-dimensional KP solution of genus 2.

For any function satisfying (4.8) and (4.1),

$$\begin{aligned}\partial_x &= \mu_1 \partial_1 + \mu_2 \partial_2, & \partial_y &= \nu_1 \partial_1 + \nu_2 \partial_2, \\ \partial_t &= \omega_1 \partial_1 + \omega_2 \partial_2,\end{aligned}\tag{5.8}$$

where  $\partial_j = \frac{\partial}{\partial \phi_j}$ ,  $j = 1, 2$ .

Now integrate  $f(x,y,0)$  over a period parallelogram (with area  $A$ ), using (4.16):

$$\frac{1}{2} (\mu_1 v_2 - \mu_2 v_1) \iint dA \cdot f(x,y,0) = \quad (5.9)$$

$$\int_0^{2\pi} d\phi_1 \int_0^{2\pi} d\phi_2 (\mu_1 \partial_1 + \mu_2 \partial_2) (\partial_x \ln \theta) = 0$$

This requirement is closely related to the normalization in (2.3).

A second constraint on the initial data is obtained by multiplying (1.8) by  $(\partial_x f)$ , and integrating over a period parallelogram. Most of the integrated terms vanish identically, and we are left with

$$\iint dA \cdot (\partial_x f(x,y,0))^3 = 0. \quad (5.10)$$

Similarly, multiplying (1.8) by  $\partial_y f$  and integrating over a period parallelogram yields

$$\iint dA \cdot (\partial_x f(x,y,0))^2 \cdot (\partial_y f(x,y,0)) = 0. \quad (5.11)$$

It is clear that an infinite sequence of necessary constraints on the initial data can be obtained in this way. It is also clear that no finite number of them will be sufficient to guarantee that  $f(x,y,0)$  is the initial data of a KP solution of genus 2. Conditions which are both necessary and sufficient are not known at this time.

Consider now the second question. Suppose  $f(x,y,0)$  is two-dimensional, is periodic in two independent directions, and is given on a period parallelogram. Further, suppose that  $f(x,y,0)$  is known to be the initial data of a KP solution of genus 2. Find  $f(x,y,t)$ .

Because  $f(x,y,0)$  defines initial data for a two-dimensional KP solution of genus 2, the initial wave pattern simply translates with some uniform velocity for all time. Once this velocity is known, the initial data in a period parallelogram plus the velocity of translation determine the solution for all time. There is no need to find the underlying Riemann matrix in this

case.

The velocity of translation can be found from (4.17) if  $(\omega_1, \omega_2)$  in (4.1) are known. One equation for  $(\omega_1, \omega_2)$  can be found by multiplying (1.8) by  $f$ , and multiplying over a period parallelogram. The result is

$$\iint dA \cdot (f_x f_t + 6ff_x^2 - f_{xx}^2 + 3f_y^2) = 0. \quad (5.12)$$

It follows from (5.1) that

$$(\mu_1 v_2 - \mu_2 v_1) f_t = (\omega_1 v_2 - \omega_2 v_1) f_x + (\mu_1 \omega_2 - \mu_2 \omega_1) f_y \quad (5.13)$$

Moreover,  $(\mu_1, \mu_2, v_1, v_2)$  all can be measured directly from the given initial data, and  $\mu_1 v_2 \neq \mu_2 v_1$  because the initial data is two-dimensional. Substituting (5.13) into (5.12) yields

$$\begin{aligned} & \iint dA (v_2 f_x^2 - \mu_2 f_x f_y) \cdot \omega_1 + \iint dA (\mu_1 f_x f_y - v_1 f_x^2) \cdot \omega_2 \\ & + (\mu_1 v_2 - \mu_2 v_1) \iint dA (6ff_x^2 - f_{xx}^2 + 3f_y^2) = 0. \end{aligned} \quad (5.14)$$

The integrals in (5.14) all can be evaluated at  $t = 0$ , so (5.14) is an algebraic equation for  $(\omega_1, \omega_2)$ .

A second equation for  $(\omega_1, \omega_2)$  can be found easily if  $(\mu_2/\mu_1)$  is rational, which we now assume. Because  $(\mu_2/\mu_1)$  can be measured from the initial data only to a finite accuracy, this assumption makes no practical limitations. Its consequence is that  $f(x, y, 0)$  is necessarily strictly periodic in  $x$  (holding  $y$  fixed). Denote this  $x$ -period by  $L$ . Because  $f(x, y, 0)$  has the form (4.8) by assumption, then for all  $(x_0, y)$ ,

$$\int_0^L dx f(x + x_0, y, 0) = 0. \quad (5.15)$$

Define

$$f(x, y; x_0) = L^{-1} \int_0^L ds \cdot sf(x + x_0 + s, y, 0) \quad (5.16)$$

$f$  is the unique anti-derivative of  $f$  with the same periodicity as  $f$ , and with zero mean in  $x$ ; i.e.,  $f$  also satisfies (5.15) and  $f = \partial_x f$ . The corresponding anti-derivative of  $f$  may be defined in a similar way.

Now multiply (1.8) by that anti-derivative of  $f$  (i.e., the second integral of  $f$ ) with the same periodicity as  $f$  and  $f$  and integrate over a period parallelogram. The final result is

$$\begin{aligned} & \iint dA \left( v_2 r^2 - \mu_2 f \delta_y \right) \cdot \omega_1 + \iint dA \left( \mu_1 f \delta_y - v_1 r^2 \right) \cdot \omega_2 \\ & + \left( \mu_1 v_2 - \mu_2 v_1 \right) \iint dA \left( 3r^3 - r_x^2 + f^2_y \right) = 0. \end{aligned} \quad (5.17)$$

The integrals may be evaluated at  $t = 0$ , so (5.14) and (5.17) are two linear algebraic equations for  $(\omega_1, \omega_2)$ . If they are linearly independent, their common solution defines  $(\omega_1, \omega_2)$ , and it completes the specification of the given KP solution of genus 2.

CONJECTURE E: (5.14) and (5.17) are linearly independent, and they define  $(\omega_1, \omega_2)$  uniquely.

It is easy to show that (5.14) and (5.17) are linearly independent in the small amplitude limit ( $d \rightarrow -\infty$ ,  $b \rightarrow -\infty$ ), but we have not established this property in general. In the cases we have tested numerically, (5.14) and (5.17) are independent, and their common solution agrees with that in (4.13).

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## Appendix A

Theorem: Let  $\phi_1$  and  $\phi_2$  be independent real variables. For fixed  $\lambda$  and  $\bar{\lambda}$  ( $0 \leq \bar{\lambda}^2 \leq \lambda^2 \leq 1/4$ ), define  $\phi_3$  and  $\phi_4$  by (4.11) on  $0 \leq \phi_1 < 2\pi$ ,  $0 \leq \phi_2 < 2\pi$ , and by periodic extension outside this rectangle. Then once each period  $\phi_3 = \text{const}$  and  $\phi_4 = \text{const}$  experience discontinuous shifts, whose size and direction are given by (4.25).

Proof: Figure 4 shows a period lattice in  $\phi_1$  and  $\phi_2$ , with two line segments of  $\phi_4 = \text{const}$ , shifted by  $\Delta\phi_4$ . We must show that  $\Delta\phi_4$  is given by (4.25). Let  $(x,y)$  denote Cartesian variables, and let  $\phi_1$  and  $\phi_2$  be related to  $(x,y)$  by (4.1) at  $t = 0$ . Let  $L$  denote the normal distance between  $\phi_2$  and  $(\phi_2 + 2\pi)$ . Then

$$L = 2\pi \left( \mu_2^2 + \nu_2^2 \right)^{-1/2} \quad (\text{A.1})$$

Let  $l_1$  denote the distance along  $(\phi_2 = \text{const})$ , between  $\phi_1$  and  $(\phi_1 + 2\pi)$ , as shown in Figure 4. Their product gives the area of the period parallelogram:

$$l_1 L = A = \frac{(2\pi)^2}{|\mu_1 \nu_2 - \mu_2 \nu_1|}, \quad (\text{A.2})$$

where we have used (4.16). This determines  $l_1$ .

The unit normal vector for any of the phase lines  $\phi_j = \text{const}$ ,  $j = 1, 2, 3, 4$  has components given by

$$\hat{n}_j = \left( \frac{\mu_j}{(\mu_j^2 + \nu_j^2)^{1/2}}, \frac{\nu_j}{(\mu_j^2 + \nu_j^2)^{1/2}} \right) \quad (\text{A.3})$$

The angle ( $\gamma$ ) between the two lines,  $\phi_2 = \text{const}$  and  $\phi_4 = \text{const}$ , is given by

$$\cos \gamma = \hat{n}_2 \cdot \hat{n}_4 = \frac{\mu_2 \mu_4 + v_2 v_4}{(\mu_2^2 + v_2^2)^{1/2} (\mu_4^2 + v_4^2)^{1/2}} ,$$

so

$$\sin^2 \gamma = 1 - \cos^2 \gamma = \frac{\lambda^2 (\mu_1 v_2 - \mu_2 v_1)^2}{(\mu_2^2 + v_2^2) (\mu_4^2 + v_4^2)} . \quad (\text{A.4})$$

Now the normal distance ( $\Delta L_4$ ) is one side of a right triangle whose hypotenuse is  $l_1$ :

$$\Delta L_4 = l_1 \sin \gamma = \frac{2\pi\lambda}{(\mu_4^2 + v_4^2)^{1/2}} .$$

But the magnitude of the wave-vector,  $\vec{\kappa}_4$  is  $(\mu_4^2 + v_4^2)^{1/2}$ , so the distance  $\Delta L_4$  corresponds to a phase shift of

$$\Delta\phi_4 = 2\pi\lambda . \quad (\text{A.5})$$

This is the desired result. The other half of (4.25) is proved by similar means.



## Appendix B

Suppose the basic period parallelogram of a fixed wave system is identified by  $\phi_1 = \text{const.}$  and  $\phi_2 = \text{const.}$  The wave crests are identified by  $\phi_3 = \text{const.}$  and  $\phi_4 = \text{const.}$ , where  $\phi_3$  and  $\phi_4$  are defined by (5.4), with  $\lambda$  and  $\bar{\lambda}$  satisfying (5.3) and (5.5). We want to show the basic period parallelogram is the only period parallelogram in which the (fixed) wave crests are related to the lines of periodicity through relations of the form (5.3)-(5.5).

Let  $\phi_a = \text{const.}$  and  $\phi_b = \text{const.}$  along the sides of another period parallelogram. If the two parallelograms enclose the same area, then  $\phi_a, \phi_b$  are related to  $\phi_1, \phi_2$  by an integer, unimodular transformation of the form

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \phi_a \\ \phi_b \end{pmatrix}$$

(B.1)

In this Appendix, let  $\sigma_n \neq 1$ , with  $\sigma_m$  and  $\sigma_n$  independent. Let  $(a,b,c,d)$  be integers with

$$ad - bc = \sigma_1.$$

(B.2)

Substituting (B.1) into (5.4) yields

$$\begin{pmatrix} \phi_3 \\ \phi_4 \end{pmatrix} = \begin{pmatrix} a - \bar{\lambda}c & b - \bar{\lambda}d \\ c - \lambda a & d - \lambda b \end{pmatrix} \begin{pmatrix} \phi_a \\ \phi_b \end{pmatrix}. \quad (\text{B.3})$$

There are two choices:

$$(i) \quad a - \bar{\lambda}c = \sigma_2, \quad d - \lambda b = \sigma_3, \quad b - \bar{\lambda}d = \bar{\Lambda}, \quad c - \lambda a = \Lambda; \quad \text{or}$$

$$(ii) \quad c - \lambda a = \sigma_2, \quad b - \bar{\lambda}d = \sigma_3, \quad d - \lambda b = \Lambda, \quad a - \bar{\lambda}c = \bar{\Lambda}.$$

For either choice,

$$\Lambda \bar{\Lambda} > 0, \quad 0 < \bar{\Lambda}^2 \leq \Lambda^2 \leq \frac{1}{4}. \quad (\text{B.4})$$

We will show that in either case one is forced back to the basic period parallelogram.

Assume (i); the analysis for (ii) is similar, and we will not repeat it. Because  $\lambda \bar{\lambda} > 0$ ,

$$c = \frac{a - \sigma_2}{\bar{\lambda}}, \quad b = \frac{d - \sigma_3}{\lambda} \quad (\text{B.5})$$

Therefore  $\Lambda = c - \lambda a = a \left[ \frac{(1 - \lambda \bar{\lambda}) - \sigma_2}{\bar{\lambda}} \right]$ ,

so that  $a = \frac{\sigma_2 + \bar{\lambda} \Lambda}{1 - \lambda \bar{\lambda}}. \quad (\text{B.6})$

Now  $|\bar{\lambda}\lambda| \leq \frac{1}{4}$ , and  $1 - \lambda\bar{\lambda} \geq 1 - \lambda^2 \geq \frac{3}{4}$ .

It follows from these relations and (B.6) that

$$1 \leq \sigma_2 a \leq \frac{5}{3}. \quad (\text{B.7})$$

But  $a$  is an integer, so from (B.7) and (B.5),

$$a = \sigma_2, \quad c = 0. \quad (\text{B.8})$$

Similarly,  $d = \sigma_3$ ,  $b = 0$ . Therefore, the transformation in (B.1) becomes

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_3 \end{pmatrix} \begin{pmatrix} \phi_2 \\ \phi_b \end{pmatrix}. \quad (\text{B.9})$$

Any choice of  $(\sigma_2, \sigma_3)$  leaves the basic period parallelogram intact. This completes the proof of uniqueness of the basic period parallelogram.

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# FIGURE CAPTIONS

Figure 1 . Cnoidal wave solution of KP. In (3.2),  $b = -3.$ ; in (3.1),  $\mu = 0.5$ ,  $\nu = -0.215$ ,  $\omega = -0.325$ .

Figure 2a. Two soliton solution of KP. In (4.23),  $\eta_3 = x + 4y - 49t$ ,  $\eta_4 = x - 4y - 49t$ ,  $\exp(A) = 16/15$ .

Figure 2b. Genus 2 solution of KP in soliton regime. In (4.6),  $b = -1$ ,  $b\lambda^2 + d = -1$ ,  $\lambda = 0.15$ ; in (4.1)  $\mu_1 = \mu_2 = 0.25$ ,  $\nu_1 = -0.1263$ ,  $\nu_2 = 0$ ,  $\omega_1 = -0.9681$ ,  $\omega_2 = -0.2695$ .

Figure 2c. Contour plot of wave shown in Figure 2b. This plot shows clearly how each wave crest experiences a phase shift from every interaction with another wave.

Figure 3 . Genus 2 solutions of KP, showing some of the variety of wave forms available.

Figure 3a. One wave is dominant.  $b = -2$ ,  $d = -3$ ,  $\lambda = 0.3$ ,  $\mu_1 = \mu_2 = 0.5$ ,  $\nu_1 = -0.3164$ ,  $\nu_2 = 0$ ,  $\omega_1 = -1.227$ ,  $\omega_2 = 0.063$ .

Figure 3b. Both wave crests are evident.  $b = -2.5$ ,  $d = -3.$ ,  $\lambda = 0.3$ ,  $\mu_1 = \mu_2 = 0.5$ ,  $\nu_1 = -0.2579$ ,  $\nu_2 = 0$ ,  $\omega_1 = -0.7360$ ,  $\omega_2 = 0.1313$ .

Figure 3c. A comparison of the waves here and in Figure 3b shows the effect of the scaling symmetry in (2.8). Here  $b = -2.5$ ,  $d = -3.$ ,  $\lambda = 0.3$ ,  $\mu_1 = \mu_2 = 0.8$ ,  $\nu_1 = -1.056$ ,  $\nu_2 = 0$ ,  $\omega_1 = -5.564$ ,  $\omega_2 = -0.5549$ .

Figure 3d. Contour plot of the wave shown in Figure 3c. Here a basic period parallelogram is shown, along with the wave crests corresponding to  $\phi_3(---)$  and  $\phi_4(-\cdot-)$ . Note that the wave crests do not lie along the directions of periodicity.

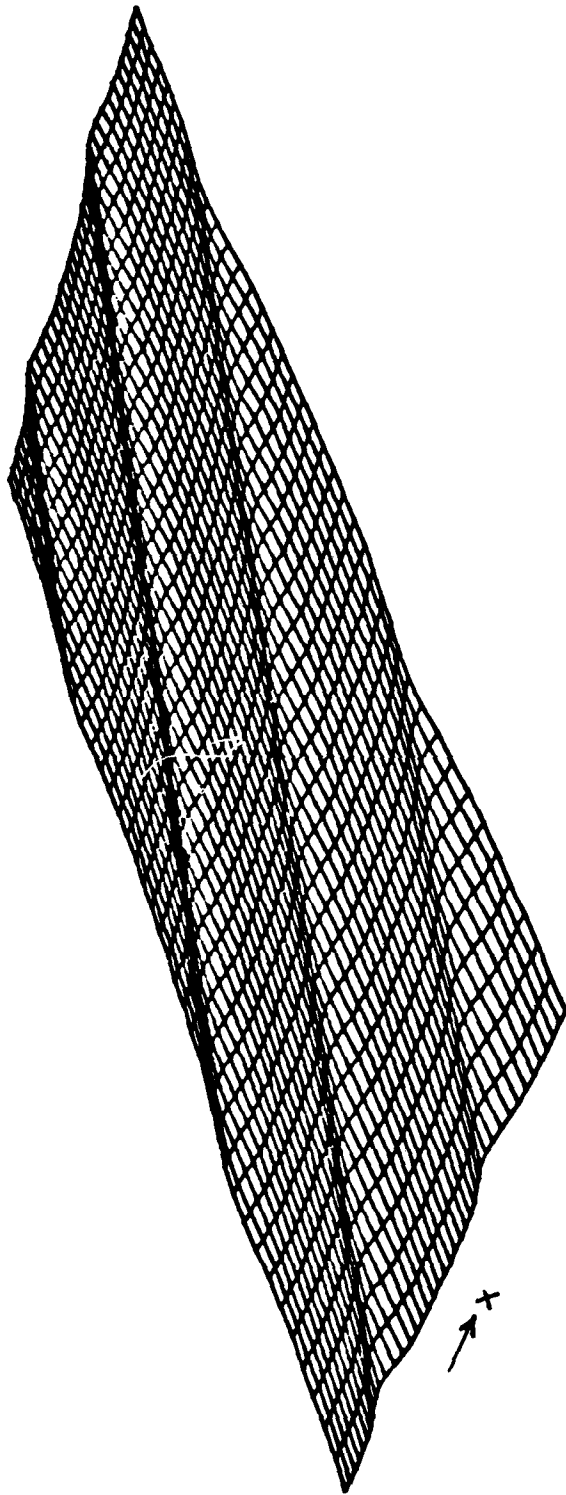
Figure 3e. Contour plot of wave pattern in the small amplitude limit.  $b = -8$ ,  $d = -8$ ,  $\lambda = 0.1$ ,  $\mu_1 = \mu_2 = 1$ ,  $\nu_1 = -1.808$ ,  $\nu_2 = 0$ ,  $\omega_1 = -8.820$ ,  $\omega_2 = 0.9837$ . In this limit (and only here) the wave crests align with the directions of periodicity.

Figure 3f. "Typical" genus 2 solution of KP, away from any limiting case.  $b = -4$ ,  $d = -3.6$ ,  $\lambda = 0.5$ ,  $\mu_1 = \mu_2 = 1$ ,  $\nu_1 = -0.7147$ ,  $\nu_2 = 0$ ,  $\omega_1 = -1.300$ ,  $\omega_2 = 0.0885$ .

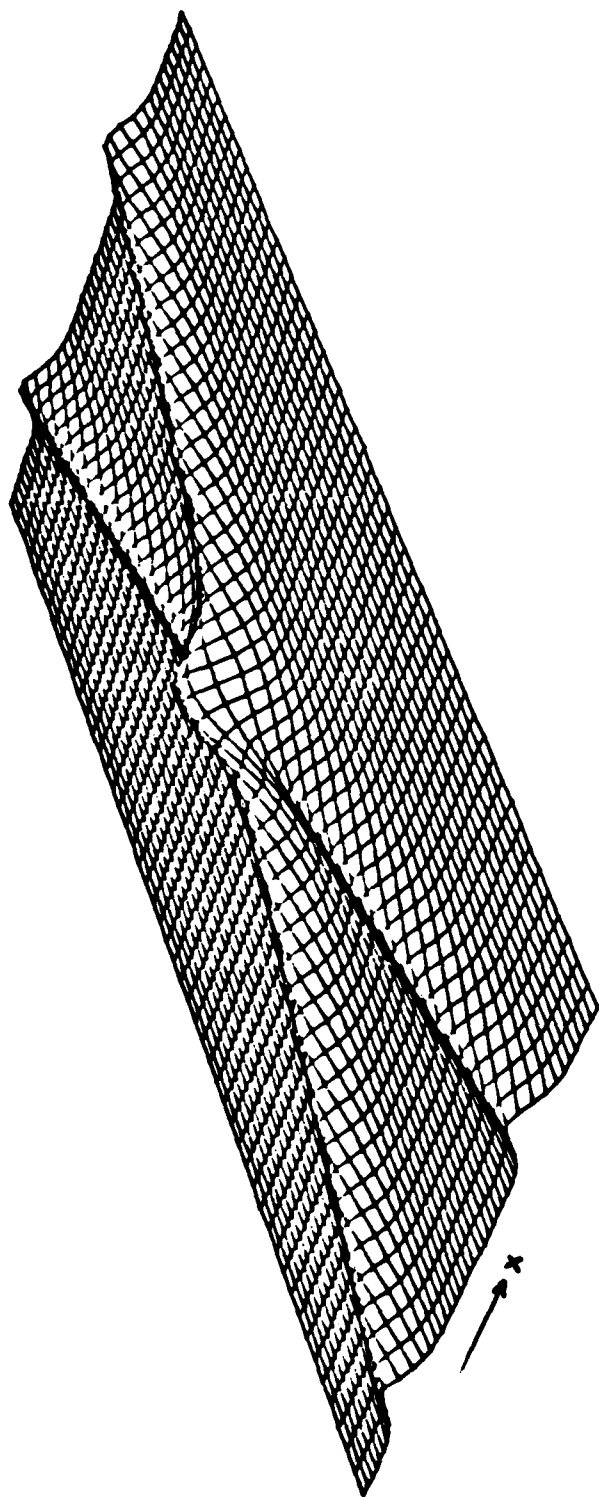
Figure 3g. Contour plot of wave pattern in Figure 3f. The wave crests and phase shifts are marked.

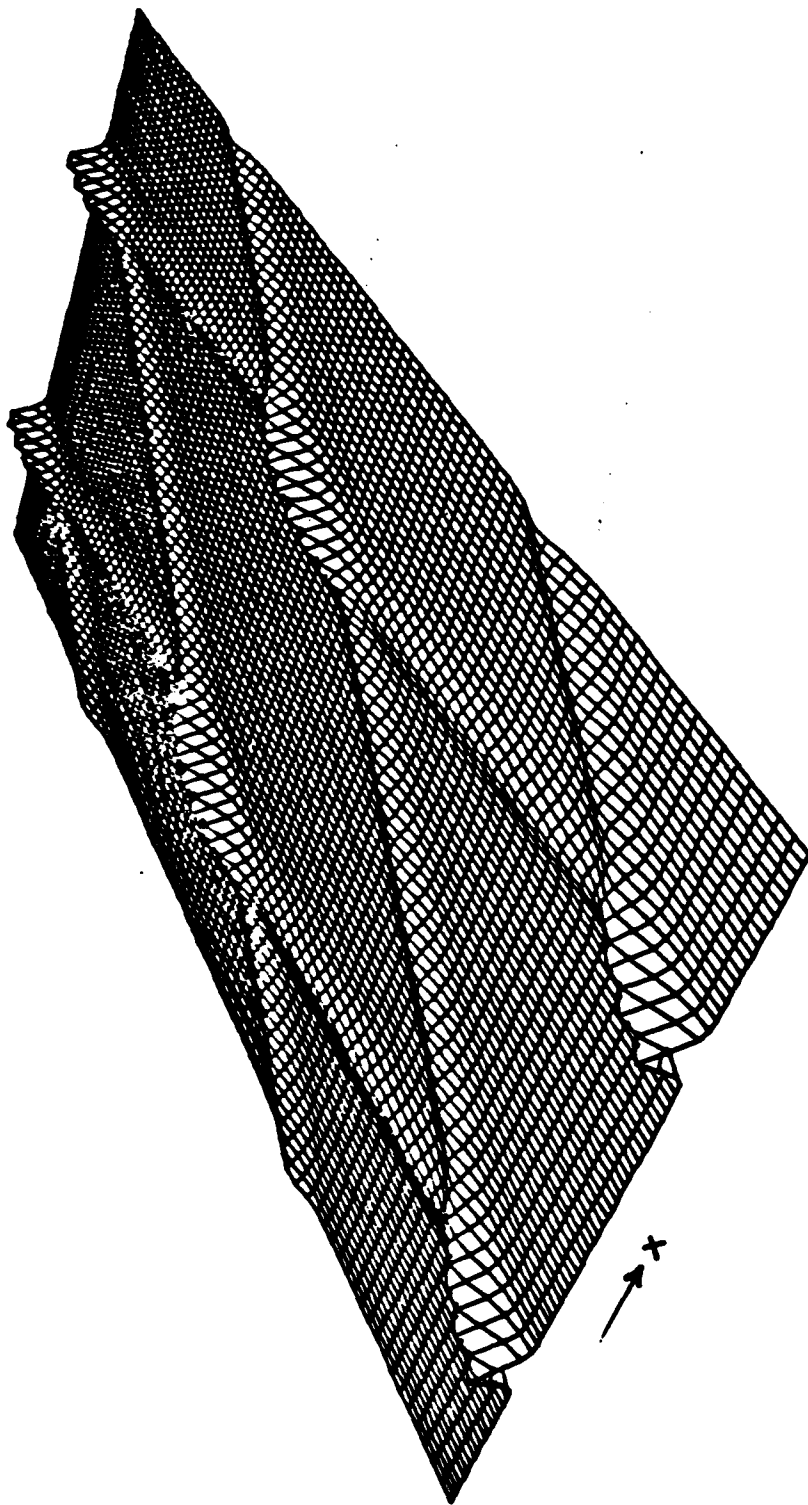
Figure 4. Period diagram of hypothetical wave pattern, showing lines of periodicity ( $\phi_1$  and  $\phi_2$ ), wave crests and phase shift. The same effects in an actual wave pattern are shown in Figure 3g.

Figure 5. A fixed period lattice supports many period parallelograms that enclose the same area. Two are shown here.

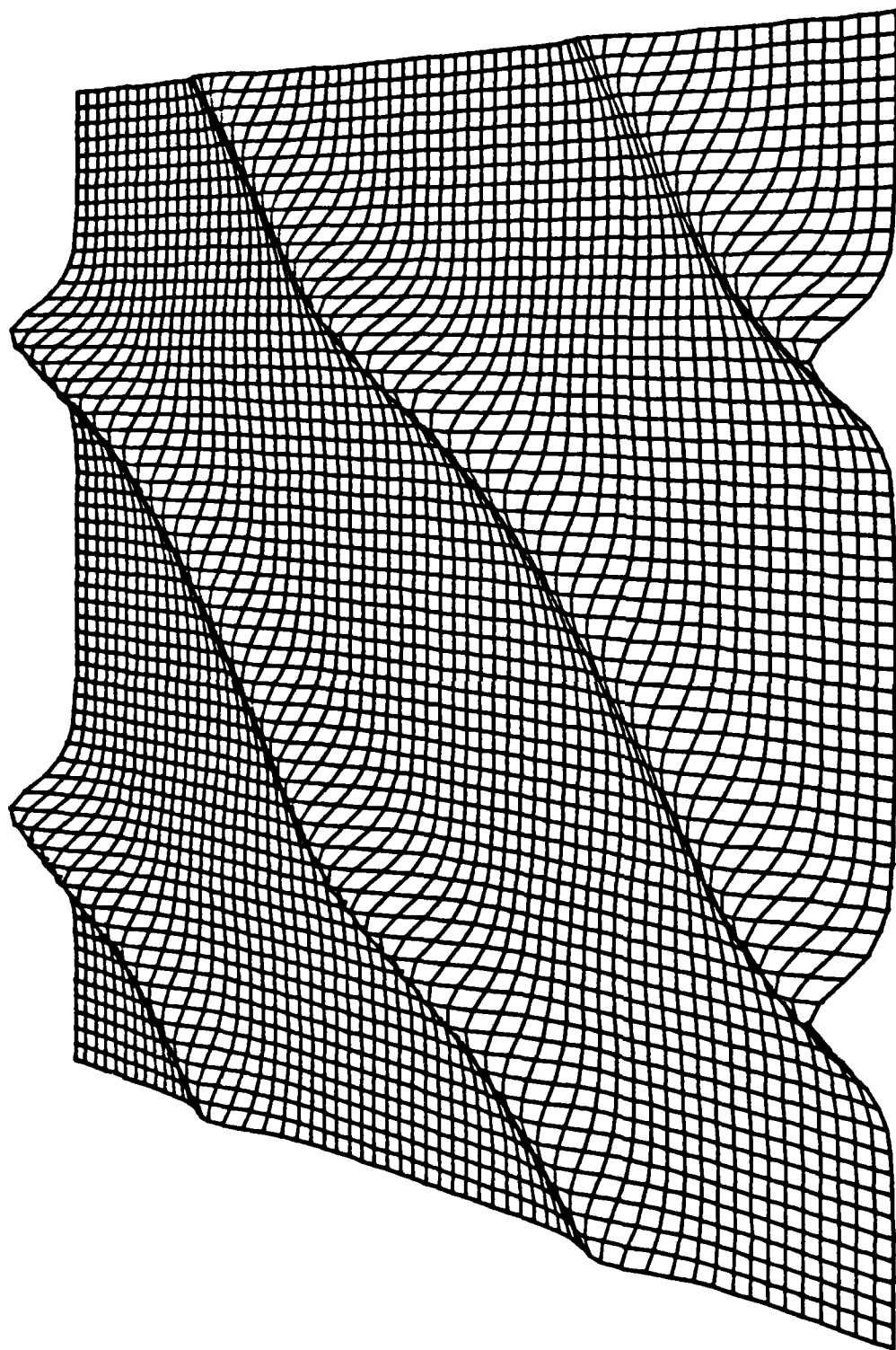




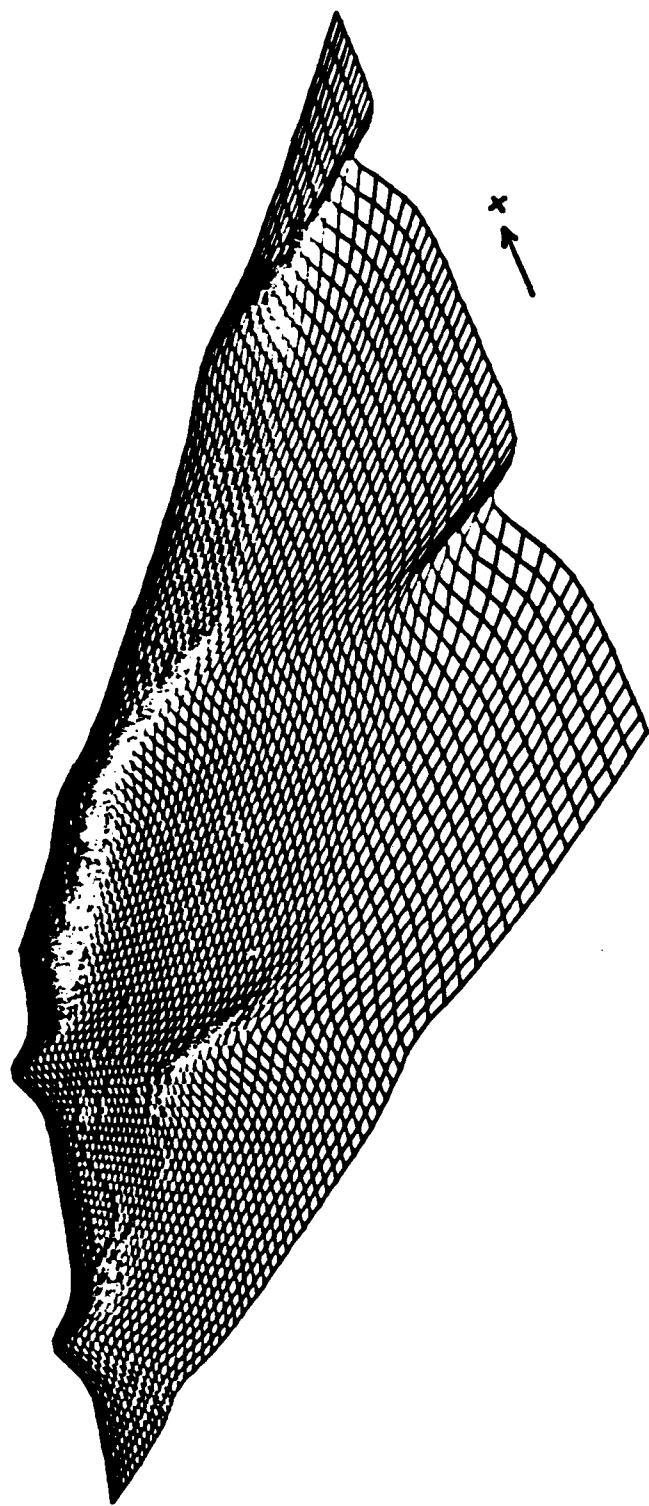


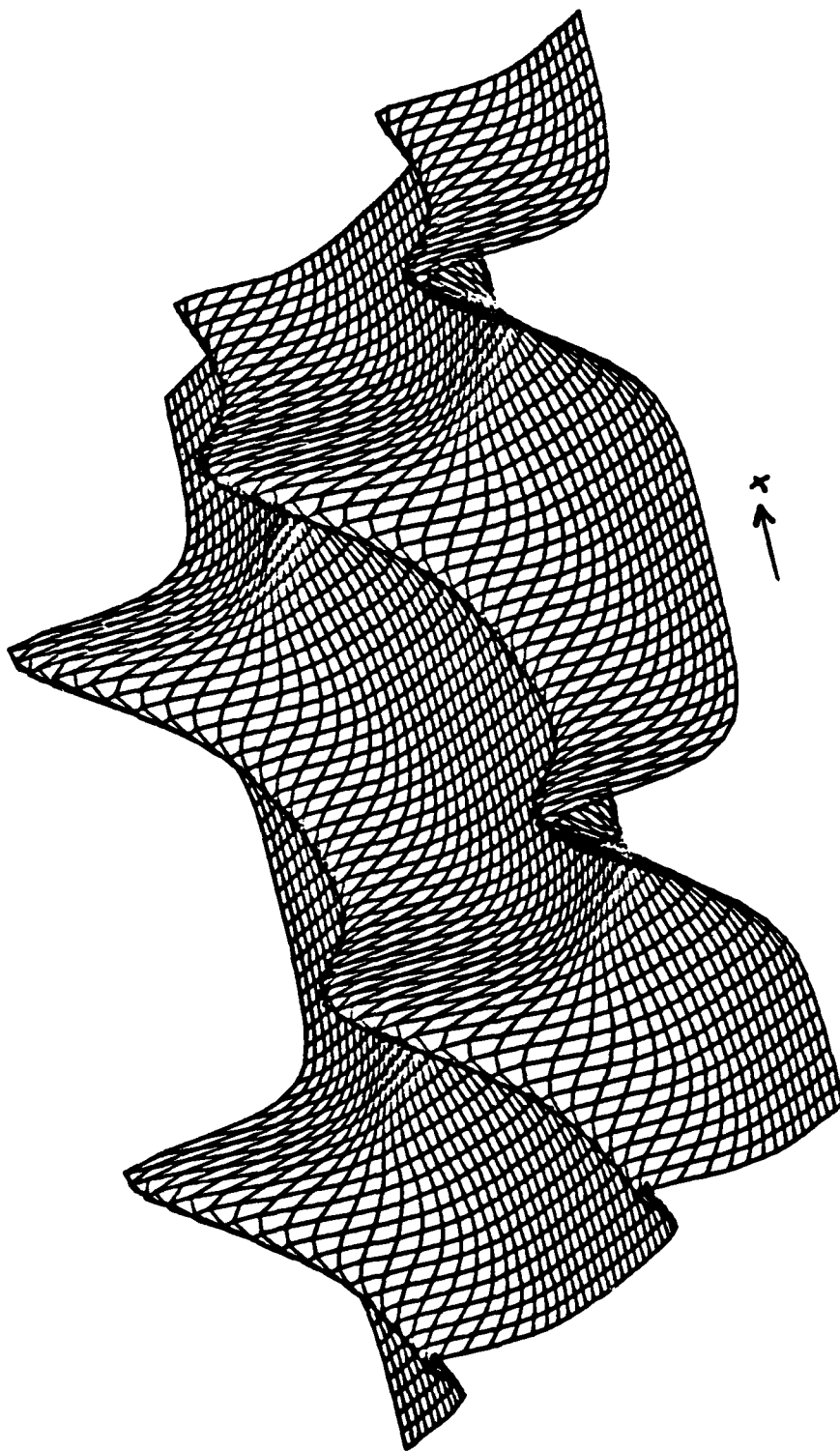


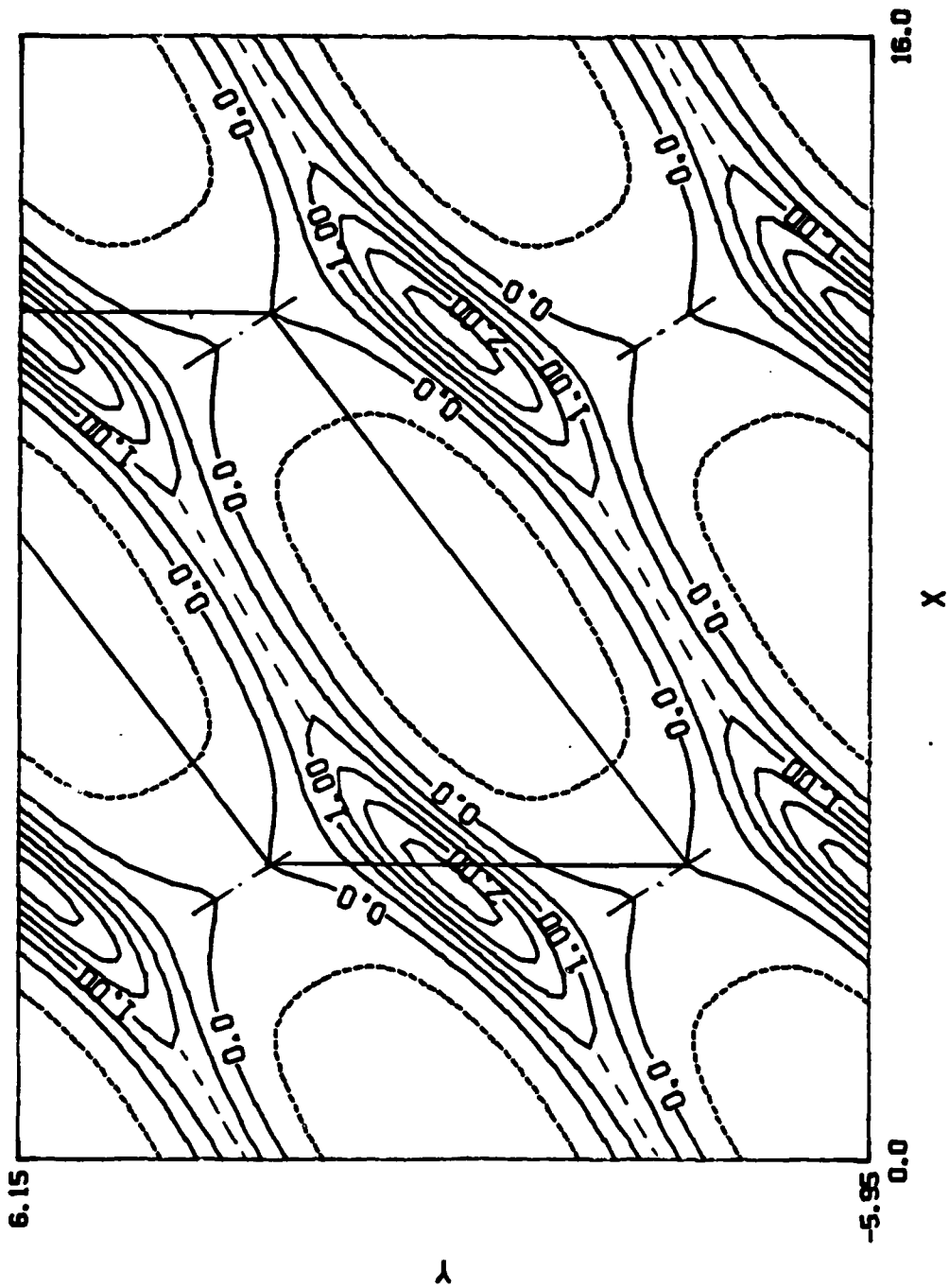


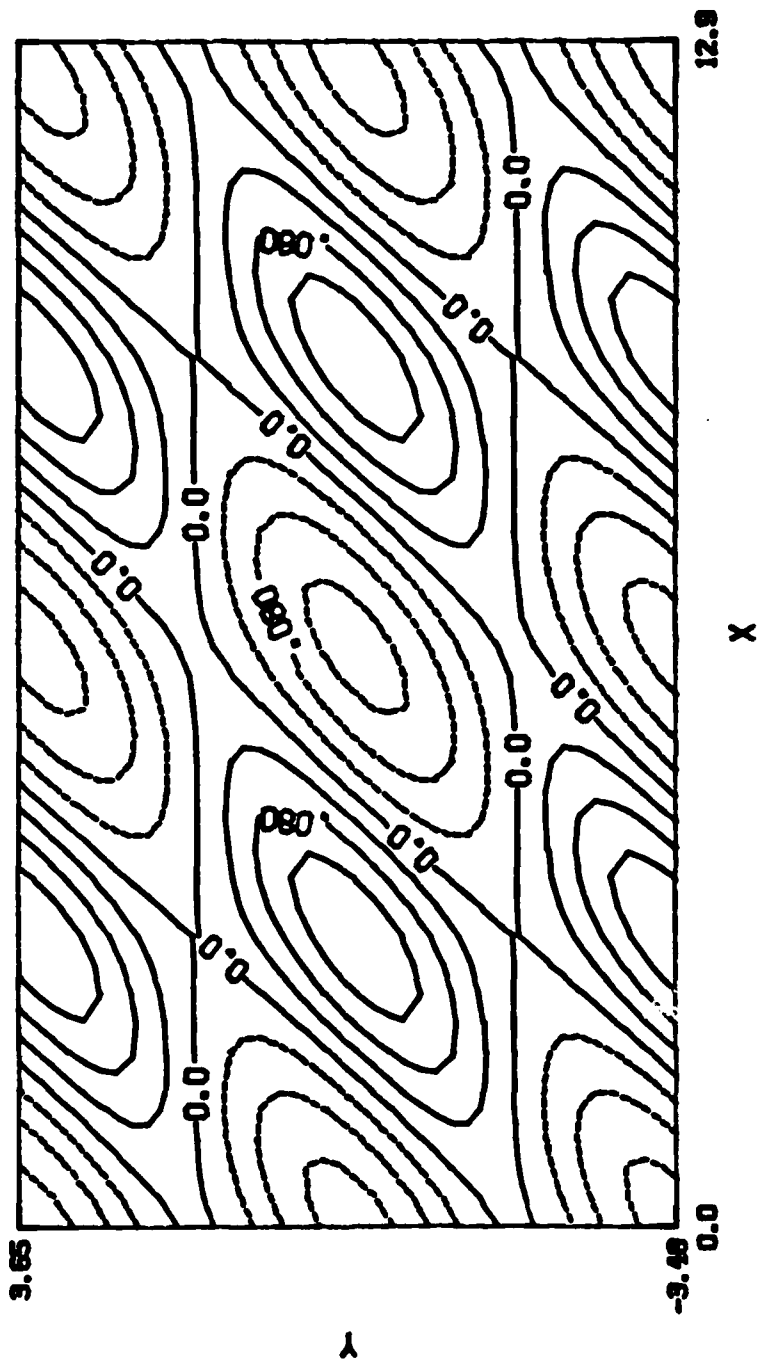


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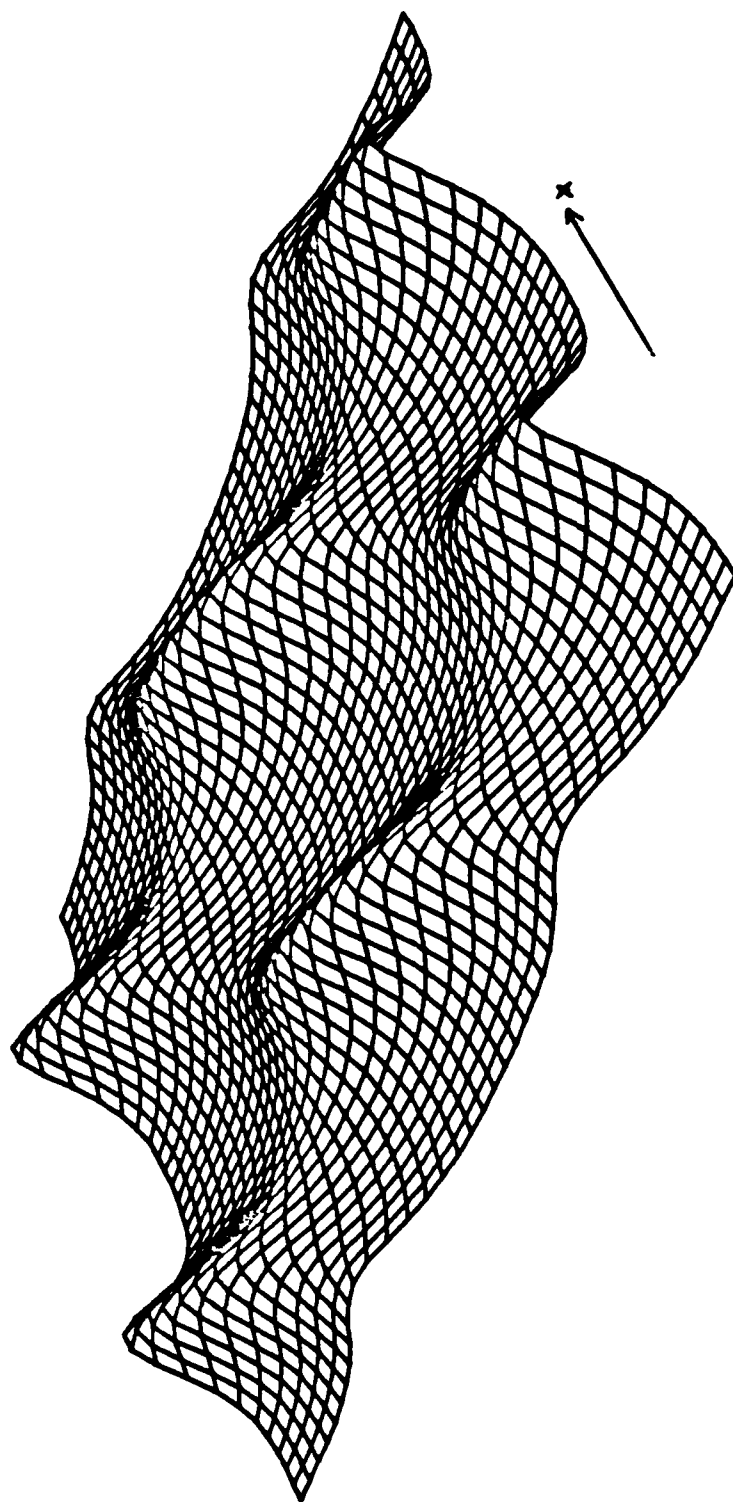




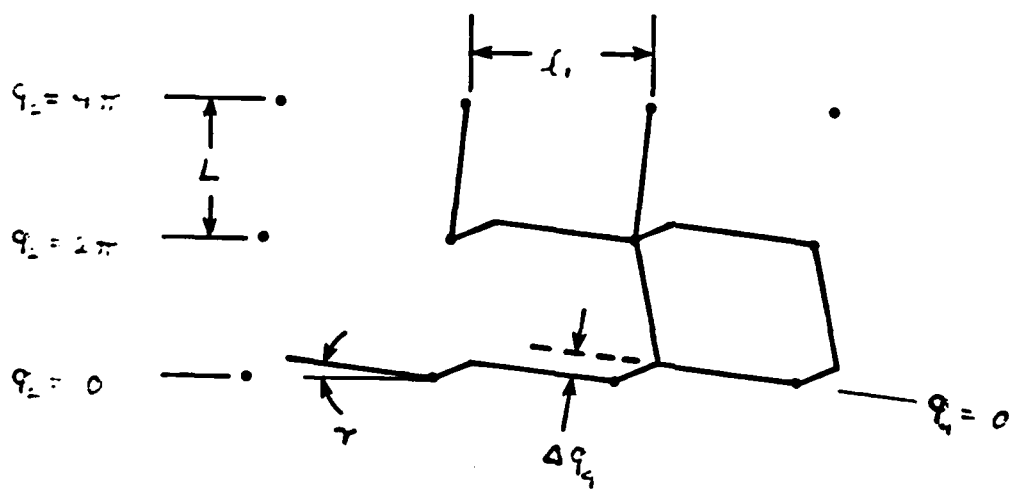


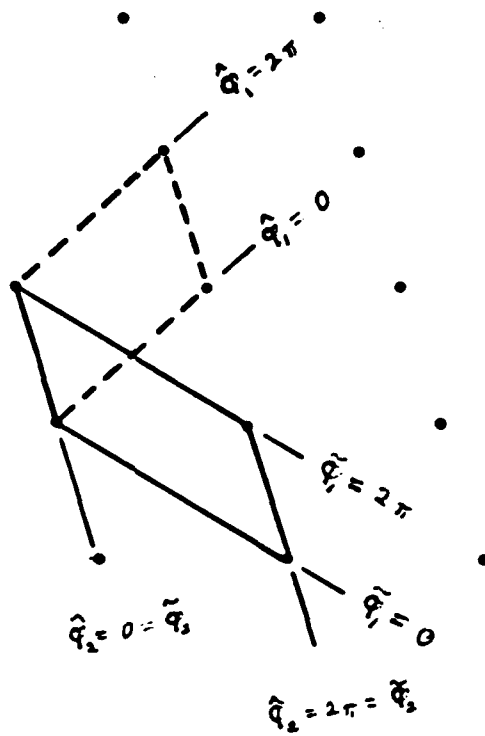












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